14.4.3. Quantifier

In Mathematics, we deal with many mathematical statements containing phrases like ***'There exists'***, ***'For every'*** or ***'For all'***. These phrases are called the ***quantifiers***. It tells us about the quantity of the variable in the statement.

Illustration 9

Consider the following statements.

p: $x + 5 > 4$ *for all* $x \in$ **N**

q: For every prime number m, $\sqrt{m}$ *is an irrational number*

r: There exists a rectangle whose all sides are equal.

The statement '*p*' means that for every natural number x, $x + 5 > 4$

The statement '*q*' means that if S denote the set of all prime numbers, then for all the members in the set S, $\sqrt{m}$ is an irrational number.

The statement '*r*' means that there is atleast one rectangle whose all sides are equal

Example 16

Identify the quantifier in the following statements and write its negation.

i. For all positive integers x, we have $x + 2 > 8$

ii. There exists a capital for every state in India.

iii. For every real number x, $x < x + 1$.

Solution

i. The quantifier is *'For all'*

Negation: *There exists a positive integer x, such that* $x + 2$ *is not greater than 8.*

or *There exists a positive integer x such that* $x + 2$ *is less than or equal to 8.*

ii. The quantifier is *'There exists'*.

Negation: *There exists a state in India which does not have a capital*

iii. The quantifier is *'For every'*

Negation: *There exists a real number x,which is not less than* $x + 1$.

or *There exists a real number x such that x is greater than or equal to* $x + 1$

SOLUTIONS TO NCERT TEXT BOOK EXERCISE 14.3

1. For each of the following compound statements first identify the connecting words and then break it into component statements.

   i. All rational numbers are real and all real numbers are not complex.

   ii. Square of an integer is positive or negative.

   iii. The sand heats up quickly in the Sun and does not cool down fast at night.

   iv. $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.

Solution

i. "And". The component statements are:

All rational numbers are real. All real numbers are not complex.

ii. "Or". The component statements are:

Square of an integer is positive. Square of an integer is negative.

iii. "And": the component statements are:

The sand heats up quickily in the sun. The sand does not cool down fast at night.

iv. "And". The component statements are:

$x = 2$ is a root of the equation $3x^2 - x - 10 = 0$

$x = 3$ is a root of the equation $3x^2 - x - 10 = 0$

2. Identify the quantifier in the following statements and write the negation of the statements.
   i. There exists a number which is equal to its square.
   ii. For every real number x, x is less than $x + 1$.
   iii. There exists a capital for every state in India.

Solution

i. "There exists".

The negation is There does not exist a number which is equal to its square.

ii. "For every".

The negation is There exists a real number x such that x is not less than $x + 1$.

iii. "There exists".

The negation is There exists a state in India which does not have a capital.

3. Check whether the following pair of statements are negation of each other. Give reasons for your answer.
   i. $x + y = y + x$ is true for every real numbers x and y.
   ii. There exists real numbers x and y for which $x + y = y + x$.

Solution

No. The negation of the statement in (i) is There exists real number x and y for which $x + y \neq y + x$, which is not the statement given in (ii).

4. State whether the *or* used in the following statements is *exclusive* or *inclusive*. Give reasons for your answer.
   i. Sun rises or Moon sets.
   ii. To apply for a driving licence, you should have a ration card or a passport.
   iii. All integers are positive or negative.

Solution

i. Exclusive   ii. Inclusive   iii. Exclusive

2. For each of the following statements, determine whether an *inclusive "Or"* or *exclusive "Or"* is used. Give reasons for your answer. **(NCERT)**

   i. To enter a country, you need a passport or a voter registration card.

   ii. The school is closed if it is a holiday or a Sunday.

   iii. Students can take French or Sanskrit as their third language.

3. Identify the type of "Or" used in the following statements and check whether the statements are true or false: **(NCERT)**

   i. $\sqrt{2}$ is a rational number or an irrational number.

   ii. To enter into a public library children need an identity card from the school or a letter from the school authorities.

   iii. A rectangle is a quadrilateral or a 5-sided polygon.

4. Identify the quantifier in the following statements and write the negation of the given statements.

   i. For every real number x, $x + 5$ is greater than 2

   ii. There exists a triangle whose sides are equal

14.5. IMPLICATIONS (Conditional Statements)

We are familiar with statements of the form **if - then, only if, if and only if,** etc. The statements with these phrases are known as **implications**. In this section, we shall discuss about conditional statements **if - then** and the biconditional statement **"if and only if"**

Conditional statement (If - then)

Let p and q be two statements then the compound statement of the form **'If p then q'** is called a conditional statement. This is denoted by $p \Rightarrow q$ (read as p implies q). Here p is the **antecedant** and q is the **consequent.**

Consider the statement

r : If ABCD is a parallelogram then AB = CD.

The component statements are p : ABCD is a parallelogram and q : AB = CD

Then r is "If p then q"

Illustration 10

Consider the implication if N is a multiple of 4, then it is a multiple of 2.

The component statements are

p : N is a multiple of 4 *q : N is a multiple of 2*

The implication if p then q is same as $p \Rightarrow q$

The statement $p \Rightarrow q$ means that the number N is a multiple of 4 is a sufficient condition to conclude that N is a multiple of 2.

That is, **$p \Rightarrow q$ is same as p is the sufficient condition for q.**

The statement $p \Rightarrow q$ means that N is a multiple of 4 only if it is a multiple of 2 or p only if q.

That is, $p \Rightarrow q$ **is same as** p **only if** q.

The statement $p \Rightarrow q$ means that when N is a multiple of 4, it is necessary that N is a multiple of 2.

That is, $p \Rightarrow q$ **is same as** q **is a necessary condition for** p.

The statement $p \Rightarrow q$ also means that when N is not a multiple of 2, then N is not a multiple of 4.

That is $p \Rightarrow q$ **is same as** $\sim q \Rightarrow \sim p$.

The different forms of ***If p*** **then** q.

i. $p \Rightarrow q$. ii. p is a sufficient condition for q. iii. *p only if q*

iv. q is a necessary condition for p. v. $\sim q$ *implies* $\sim p$

Example 17

Rewrite the following statement with "if-then" in five different ways conveying the same meaning. **(NCERT)**

If a natural number is odd, then its square is also odd.

Solution

i. A natural number is odd implies that its square is odd.

ii. For the square of a natural number to be odd, it is sufficient that the number is odd

iii. A natural number is odd only if its square is odd.

iv. For a natural number to be odd it is necessary that its square is odd.

v. If the square of a natural number is not odd, then the natural number is not odd.

Example 18

For the given statements identify the necessary and sufficient conditions.

t: If you drive over 80 km per hour, then you will get a fine. **(NCERT)**

Solution

Let p and q be the component statements.

p : *You drive over 80 km per hour.*

q : *You will get a fine.*

In the conditional statement

"if p then q", the necessary condition is p and the sufficient condition is q.

$\therefore$ Necessary condition : You drive over 80 km per hour

Sufficient condition : You will get a fine.

Truth value of Implication

Illustration 11

i. Let p : *18 is divisible by 6*

q : *18 is divisible by 2*

Both p and q are true. We also observe that $p \Rightarrow q$ is true.

ii. Let p : *19 is divisible by 6*

q : *19 is divisible by 2*

Both p and q are false. We observe that $p \Rightarrow q$ is true.

iii. Let p : *18 is divisible by 6*

q : *18 is not divisible by 2*

Here p is true and q is false. Also $p \Rightarrow q$ is false.

iv. Let p : *14 is divisible by 6.*

q : *14 is divisible by 2*

Here p is false and q is true. We observe that $p \Rightarrow q$ is true.

This illustration leads us to the following rule.

Rule for compound statement $p \Rightarrow q$

The implication *"if p then q"* is always true except when p is true and q is false.

Example 19

Write the component statement of each of the following statements. Also check whether the statements are true or not.

i. If Δ ABC is equilateral, then it is isosceles. **(September 2010)**

ii. If x and y are integers, then xy is an irrational number. **(NCERT)**

Solution

i. The component statements are given by

p: Δ ABC is equilateral $\quad$ q: Δ ABC is isosceles.

Since an equilateral triangle is isosceles, the given *compound statement* is true.

STUDY TIP

$p \Rightarrow q$ is same as $\sim p$ or q.

ii. The component statements are given by

p: x and y are integers $\quad$ q: xy is an irrational number.

Since the product of two integers is an integer it is not an irrational number, the given *compound statement* is false.

14.5.1. Contrapositive and Converse

In this section we discuss how to form the contrapositive and converse of a conditional statement $p \Rightarrow q$ (If p then q)

Contrapositive of the implication if p then q or $p \Rightarrow q$

If p and q are two statements, then contrapositive of the implication *if p then q* is ***if ~q then ~p***

Example 20

Write down the contrapositive statements of the following statements

i. If a number is divisible by 9, then it is divisible by 3 **(September 2010)**

ii. If a triangle is equilateral, then it is isosceles. **(March 2010)**

Solution

Contrapositive of the statements are

i. If a number is not divisible by 3, then it is not divisible by 9

ii. If a triangle is not isosceles, then it is not equilateral.

Example 21

Write the contrapositive and the converse of the statement. "If a triangle is not isosceles, then it is not equilateral". **(March 2015)**

Solution

Contrapositive of the statement:

If a triangle is equilateral, then it is isosceles.

Converse of the statement:

If a triangle is not equilateral, then it is not isosceles.

Converse of the implication if p then q or $p \Rightarrow q$

The converse of the implication *if p then q* is obtained by interchanging the *antecedant* and *consequent* of the given statement. The converse of the implication *if p then q* is ***if q then p***.

Example 22

Write the converse of the following statements.

i. If a number n is even, then n^2 is even. **(NCERT, March 2011, August 2014)**

ii. If r is the radius of a circle, then its area is πr^2

iii. If you do all the exercises in this book, then you will get an 'A grade in the class'.

iv. If a triangle is equilateral, then all the angles are equal.

v. If x is an integer and x^2 is even, then x is also even. **(March 2012)**

Solution

The converse of the statements are

i. If a number n^2 is even, then n is even

ii. If the area of a circle is πr^2, then its radius is r

iii. If you get an 'A grade' in the class, then you have done all 'exercises of this book'.

iv. If all the angles of a triangle are equal, then it is an equilateral triangle.

v. If the integer x is even, then x^2 is even.

Biconditional or *if and only if* implication

If p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called an if and only if implication denoted as $p \Leftrightarrow q$.

The equivalent forms of $p \Leftrightarrow q$ are as follows.

i. p if and only if q

ii. q if and only if p

iii. p is necessary and sufficient condition for q and vice-versa.

The following are examples of *if and only if* implications

i. A quadrilateral is equiangular if and only if it is a rectangle.

ii. You will get an 'A grade' if and only if you do all the homework regularly

Example 23

Combine the following statements using *if and only if.*

p: If a rectangle is a square, then all its four sides are equal

q: If all the four sides of a rectangle are equal, then the rectangle is a square.

Solution

A rectangle is a square if and only if all its four sides are equal.

Example 24

Using the words "necessary and sufficient" rewrite the statement.

"The integer n is odd if and only if n^2 is odd". **(NCERT)**

Solution

The necessary and sufficient condition that the integer n be odd is n^2 must be odd.

Rule for biconditional statement $p \Leftrightarrow q$

The biconditional statement $p \Leftrightarrow q$ (p *if and only if* q) is true when the component statements have the same truth value.

SOLUTIONS TO NCERT TEXT BOOK EXERCISE 14.4

1. Rewrite the following statement with "if-then" in five different ways conveying the same meaning. If a natural number is odd, then its square is also odd.

Solution

Refer Example 17

2. Write the contrapositive and converse of the following statements.

i. If x is a prime number, then x is odd. **(March 2013, September 2013)**

ii. If the two lines are parallel, then they do not intersect in the same plane.

iii. Something is cold implies that it has low temperature.

iv. You cannot comprehend geometry if you do not know how to reason deductively.

v. x is an even number implies that x is divisible by 4.

Solution

i. The contrapositive is If a number x is not odd, then x is not a prime number.

The converse is If a number x in odd, then it is a prime number.

ii. The contrapositive is If two lines intersect in the same plane, then they are not parallel

The converse is If two lines do not interesect in the same plane, then they are parallel

iii. The contrapositive is If something is not at low temperature, then it is not cold

The converse is If something is at low temperature, then it is cold

iv. The contrapositive is If you know how to reason deductively, then you can comprehend geometry.

The converse is If you do not know how to reason deductively, then you cannot comprehend geometry.

v. This statement can be written as "If x is an even number, then x is divisible by 4".

The contrapositive is, If x is not divisible by 4, then x is not an even number.

The converse is, If x is divisible by 4, then x is an even number.

3. Write each of the following statements in the form "if-then"
   i. You get a job implies that your credentials are good.
   ii. The Bannana trees will bloom if it stays warm for a month.
   iii. A quadrilateral is a parallelogram if its diagonals bisect each other.
   iv. To get an A^+ in the class, it is necessary that you do all the exercises of the book.

Solution

i. If you get a job, then your credentials are good.
ii. If the banana tree stays warm for a month, then it will bloom.
iii. If diagonals of a quadrilateral bisect each other, then it is a parallelogram.
iv. If you get A^+ in the class, then you do all the exercises in the book.

4. Given statements in (a) and (b). Identify the statements given below as contrapositive or converse of each other.
   a. If you live in Delhi, then you have winter clothes.
      i. If you do not have winter clothes, then you do not live in Delhi.
      ii. If you have winter clothes, then you live in Delhi.
   b. If a quadrilateral is a parallelogram, then its diagonals bisect each other.
      i. If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.
      ii. If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Solution

a. i. Contrapositive ii. Converse
b. i. Contrapositive ii. Converse