14.6. VALIDATING STATEMENTS

In this section, we shall discuss the validity of statements. Checking the validity of a statement means to check whether it is true or not. The validity of a statement depends upon (i) the special words like *'and'*, *'or'* (ii) the implications *'if ..... then' if and only if* (iii) the quantifiers *for all, there exists* etc., that appear in the given statement. The following techniques are used to check the validity of statements.

i. Validity of statements with 'and'

If p and q are two mathematical statements, then in order to show that the statement *'p and q'* is true, we follow the following steps

Step i. Show that p is true

Step ii. Show that q is true

Example 25

Check the validity of the statement, 40 is a multiple of 5 and 4.

Solution

The component statements are

p: 40 is a multiple of 5 $\quad$ q: 40 is a multiple of 4

Here p and q are true.

Therefore the compound statement *p and q* is a valid statement.

ii. Validity of statements with 'or'

If p and q are two mathematical statements, then in order to show that the statement *p or q* is true, we must show that atleast one of p, q is true.

Example 26

Consider the following statements

p: 40 is a multiple of 5 $\quad$ *q: 40 is a multiple of 7*

Write the compound statement using *or* and check its validity.

Solution

The compound statement is *40 is a multiple of 5 or 7.*

p : 40 is a multiple of 5 is true. q : 40 is a multiple of 7 is false.

Hence the compound statement is valid.

Example 27

Given below are two statements

p : *25 is a multiple of 5.*

q : *25 is a multiple of 8.*

Write the compound statements connecting these two statements with "And" and "Or". In both cases check the validity of the compound statement.

Solution

Here p is true and q is false.

The compound statement with *and* is ***25 is a multiple of 5 and 8***

This is a false statement, since q is false.

The compound statement with *or* is ***25 is a multiple of 5 or 8***

This is true statement, since p is true.

iii. **Validity of statements with 'if ......then'**

If p and q are two mathematical statements, then to prove the validity of the statement "*if p then q*", we may use any one of the following methods.

1. **Direct Method**

   Assume that p is true. Then prove that q must be true.

2. **Contrapositive Method**

   Assume that q is false. Then prove that p is false

3. **Contradiction Method**

   Step i : Assume that p is true and q is false.

   Step ii : Obtain a contradiction from step i

STUDY TIPS

Direct method : $p \Rightarrow q$

Contrapositive method : $\sim q \Rightarrow \sim p$

Contradiction method :

Assume p is true and q is false, obtain a contradiction.

Example 28

Check whether the statement *If x and y are odd integers, then xy is an odd integer* is true or false by i. direct method

ii. contrapositive method

iii. contradiction method

(August 2009, March 2010)

Solution

The component statements are

p : x and y are odd integers q : xy is an odd integer

Therefore the given statement is *if p then q*

Direct Method

Let p be true $\Rightarrow x$ and y are odd integers

$\Rightarrow x = (2m + 1)$ and $y = (2n + 1)$ for integers m and n

$\Rightarrow xy = (2m + 1)(2n + 1) \quad \Rightarrow xy = 4mn + 2m + 2n + 1$

$\Rightarrow xy = 2(2mn + m + n) + 1 \Rightarrow xy$ is an odd integer,($\because$ *2mn+m+n is an integer)*

$\Rightarrow q$ is true

Thus p is true implies q is true. Therefore *if p then q* is a true statement.

Contrapositive Method

Let q be not true $\Rightarrow xy$ is an even integer

$\Rightarrow$ either x is even or y is even or both x and y are even

$\Rightarrow p$ is not true.

Thus q is false implies p is false. Therefore *if p then q* is a true statement.

Contradiction Method

Assume that p is true and q is false. This means that $\sim q$ is true. i.e., $\sim q$: xy is even

p is true $\Rightarrow x$ and y are odd integers

$\Rightarrow x = (2m + 1)$ and $y = (2n + 1)$ for some integer m and n

$\Rightarrow xy = (2m + 1)(2n + 1)$

$\Rightarrow xy = 2(2mn + m + n) + 1 \qquad \Rightarrow xy$ is odd, which is a contradiction

Therefore *if p then q* is a true statement.

Example 29

Consider the statement.

"If $(3n + 2)$ is an odd natural number, then n is an odd natural number".

i. Write its contrapositive.

ii. Check whether the given statement is true by the contrapositive method. **(October 2011)**

Solution

i. If n is not an odd natural number, then $(3n + 2)$ is not an odd natural number. **OR**

If n is an even natural number, then $(3n + 2)$ is an even natural number.

ii. Let $p : (3n + 2)$ is an odd natural number.

$q : n$ is an odd natural number.

Assume that q is false.

ie., n is an even natural number.

$\Rightarrow n = 2k$, for some integer k.

$\Rightarrow 3n + 2 = 3(2k) + 2$

$= 2\,[3k + 1]$ is an even integer.

ie., p is false.

$\therefore$ The given statement is true.

iv. **Validity of statements with if and only if**

In order to prove the validity of the statement '*p if and only if q*', we proceed as follows.

Step i : **Show that if p is true, then q is true**

Step ii: **Show that if q is true, then p is true.**

Example 30

Consider the statement *The integer x is odd if and only if x^2 is odd.* Check whether the statement is true. **(NCERT)**

Solution

The component statements are p: integer x is odd q: x^2 is odd

$\therefore$ The given statement is *p if and only if q*

Case i: Check the validity of *if p then q.*

The statement is given by *If the integer x is odd, then x^2 is odd.* We use the direct method to prove the validity of the statement.

Let us assume that x is odd.

$\Rightarrow x = (2m + 1)$ for some integer m $\Rightarrow x^2 = (2m + 1)^2$

$\Rightarrow x^2 = 4m(m + 1) + 1$ $\Rightarrow x^2$ is an odd integer

Thus x is odd implies x^2 is odd. Therefore *if p then q* is a true statement.

Case ii : Check the validity of *if q then p*

The statement is given by *If x is an integer and x^2 is odd, then x is odd.* We use the contrapositive method to prove the validity of this statement.

Assume that x is even $\Rightarrow x = 2m$ for some integer m $\Rightarrow x^2 = 4m^2$

$\Rightarrow x^2$ is an even integer $\Rightarrow x^2$ is not an odd integer

Thus x is not odd implies x^2 is not odd. Therefore *if q then p* is a true statement.

$\therefore$ *p if and only if q* is true.

v. **Validity of statement by contradiction**

To check whether a statement p is true or not, we assume that p is not true. Then we arrive at some result which contradicts our assumption. Therefore, we conclude that p is true.

Example 31

Verify by the method of contradiction that $\sqrt{2}$ is irrational **(March 2009, 2011)**

Solution

To prove $\sqrt{2}$ is irrational.

Assume that $\sqrt{2}$ is not irrational. That is $\sqrt{2}$ is rational.

$\therefore \sqrt{2} = \frac{a}{b}$, where a and b are integers having no common factor

Squaring, we get, $2 = \frac{a^2}{b^2}$ or $a^2 = 2b^2$

i.e. 2 divides a^2 $\Rightarrow$ 2 divides a *since 2 is a prime number*

Let $a = 2k$ for some integer k

$\Rightarrow a^2 = 4k^2$ $\Rightarrow 2b^2 = 4k^2$

$\Rightarrow b^2 = 2k^2$ $\Rightarrow$ 2 divides b^2 $\Rightarrow$ 2 divides b

Thus we get 2 divides a and 2 divides b. That is 2 is a common factor to both a and b which contradicts the fact that a and b have no common factor. Hence our assumption that $\sqrt{2}$ is rational is false. Therefore $\sqrt{2}$ is irrational

Validity of statements by using a counter example

In order to show that a statements is false, we may give an example of a situation where the statement is not valid. Such an example is called a counter example.

Example 32

By giving a counter example, show that the following statement *'if n is an odd integer, then n is prime'* is false. **(NCERT)**

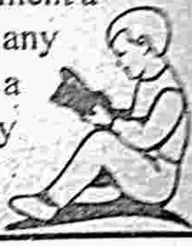
Solution

We observe that $n = 15$ is an odd integer, which is not prime. So the statement is not valid.

$\therefore n = 15$ is a counter example.

STUDY TIPS

In Mathematics, to disprove a statement a counter example is sufficient. But any number of examples in favour of a statement do not prove the validity of the statement.



SOLUTIONS TO NCERT TEXT BOOK EXERCISE 14.5

1. Show that the statement

p: If x is a real number such that $x^3 + 4x = 0$, then x is 0 is true by

i. direct method   ii. method of contradiction   iii. method of contrapositive

Solution

The component statements

$p : x$ is a real number such that $x^3 + 4x = 0$ $q : x = 0$

Then the statement is 'If p then q'

i. Direct Method

Assume p is true.

p is true $\Rightarrow x^3 + 4x = 0$ $\Rightarrow x(x^2 + 4) = 0$ $\Rightarrow x = 0$ or $x^2 + 4 = 0$

$\Rightarrow x = 0$, since $x^2 + 4 \neq 0$ $\Rightarrow q$ is true

Thus p is true implies q is true.

Hence the statement 'If p then q is true or valid.

ii. **Method of contradiction**

Assume p is true and q is false. This means that $\sim q$ is true.

That is $\sim q : x \neq 0$

$x \neq 0 \Rightarrow x^2 \neq 0 \Rightarrow x(x^2+4) \neq 0 \Rightarrow p$ is false

This is a contradiction

Hence the statement "If p then q" is true or valid.

iii. **Method of contrapositive**

Assume that q is false.

q is false $\Rightarrow x \neq 0 \Rightarrow x(x^2+4) \neq 0$ *since $x^2 + 4$ is non zero for any real x.*

$\Rightarrow x(x^2+4) \neq 0$ since $x^2 + 4 \neq 0$ and $x \neq 0 \Rightarrow p$ is false

Thus q is false implies p is false.

Hence the statement "If p then q" is true.

2. Show that the statement "For any real numbers a and b, $a^2 = b^2$ implies that $a = b$" is not true by giving a counter-example.

Solution

Let us take two real numbers a and b as $a = 2$, $b = -2$

Then $a^2 = (2)^2 = 4$ and $b^2 = (-2)^2 = 4$

So $a^2 = b^2$

But $a \neq b$ since $2 \neq (-2)$

So the given statement is not a true statement.

3. Show that the following statement is true by the method of contrapositive. **(March 2012)**

p: If x is an integer and x^2 is even, then x is also even.

Solution

The component statements are

$p : x$ is an integer and x^2 is even $\quad q : x$ is an even integer.

The given statement is 'If p then q'

Let us assume q is not true.

$\sim q \Rightarrow x$ is not an even integer $\Rightarrow x$ is an odd integer

$\Rightarrow x = 2m + 1$ for some interger m. $\Rightarrow x^2 = (2m+1)^2 \Rightarrow x^2 = 4(m^2 + m) + 1$

$\Rightarrow x^2$ is odd integer $\Rightarrow \sim p$ is true

Thus $\sim q \rightarrow \sim p$

Hence the given statement 'If p then q' is true or valid.

4. By giving a counter example, show that the following statements are not true.

i. p : If all the angles of a triangle are equal, then the triangle is an obtuse angled triangle.

ii. q : The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.

Solution

i. Consider an equilateral triangle. Then all the angles of the triangle are 60°. So it is not an obtuse angled triangle. So by this counter example, we can show that this statement is false.

ii. Let $x = 1$ then x is lying between 0 and 2.

Also $x^2 - 1 = (1)^2 - 1 = 1 - 1 = 0$

i.e., $x = 1$ is lying between 0 and 2 and $x^2 - 1 = 0$.

So by this counter example, we can show that this statement is false.

5. Which of the following statements are true and which are false? In each case give a valid reason for saying so.

i. p: Each radius of a circle is a chord of the circle.

ii. q: The centre of a circle bisects each chord of the circle.

iii. r: Circle is a particular case of an ellipse.

iv. s: If x and y are integers such that $x > y$, then $-x < -y$.

v. t : $\sqrt{11}$ is a rational number.

Solution

i. False. The radius of a circle is the line segment joining centre of a circle to any point on it, where as chord joins two points of a circle.

ii. False. Consider a chord other than the diameter. Then the chord is not bisected by the centre of the circle.

iii. True. In the equation of ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b} = 1$, put $a = b$ then we get $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \Rightarrow x^2 + y^2 = a^2$, the equation of a circle.

iv. True. Let $x > y$. Multiplying by (-1), we get $-x < -y$.

v. False. Let $\sqrt{11}$ be rational number

Then $\sqrt{11} = \frac{a}{b}$, where a, b are integers with no common factors.

$\Rightarrow 11 = \frac{a^2}{b^2} \Rightarrow a^2$ is divisible by 11 $\Rightarrow a$ is divisible by 11 $\Rightarrow a = 11k$ for some k

$\Rightarrow a^2 = (11k)^2 \Rightarrow 11b^2 = 121k^2 \Rightarrow b^2 = 11k^2 \Rightarrow b^2$ is divisible by 11

$\Rightarrow b$ is divisible by 11

Thus we get both a and b are divisible by 11, which is a contradiction to the fact that a and b have common factors.

Hence our assumption : $\sqrt{11}$ is rational is false.

$\therefore \sqrt{11}$ is irrational

Alternate Method

We know that 11 is a prime number since it has factors 1 and 11.

$\therefore \sqrt{11}$ is an irrational number since the square root of a prime number is an irrational number.