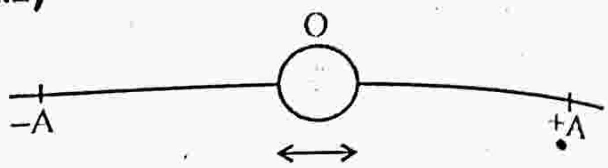
14.3 Simple Harmonic Motion (SHM)

A particle is said to execute SHM if its acceleration is proportional to displacement and is always directed towards a fixed point.



A particle vibrating back and forth about the origin of X-axis, between the limits +A and –A

Fig. 3

The block attached to the spring (see fig. 2(b)) is assumed as a particle of mass 'm'. During its back and forth motion along the x-axis, the turning points are at +A and – A w.r.to the origin O as shown in fig.3. The speed is zero at the turning points and maximum at origin. This is a special type of periodic motion. The positions are now plotted against time. At equal intervals we get the graph shown in Fig. 4.

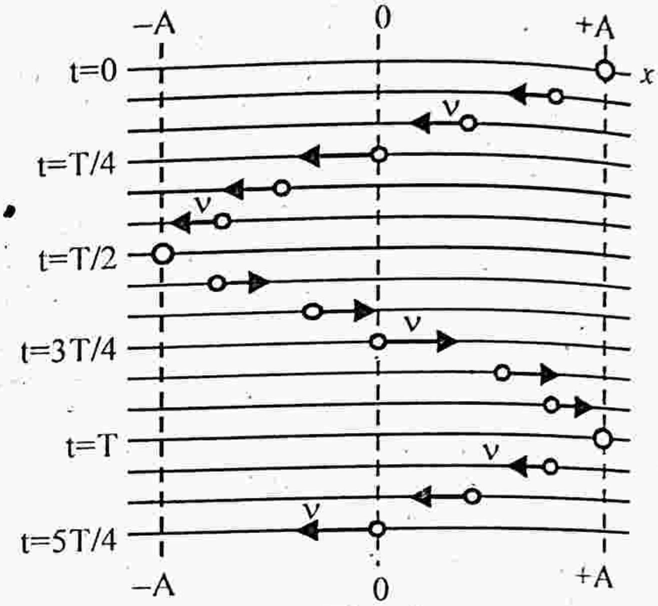


Fig. 4

The variation shown in fig.5 is identical with the cosine curve. Therefore the displacement can be taken as $x(t) = A\cos\omega t$ or more generally $x(t) = A\cos(\omega t + \phi)$ .... (1) in which A, ω and ϕ are constants.

Instantaneous displacement related with other quantities is shown in fig. 6.

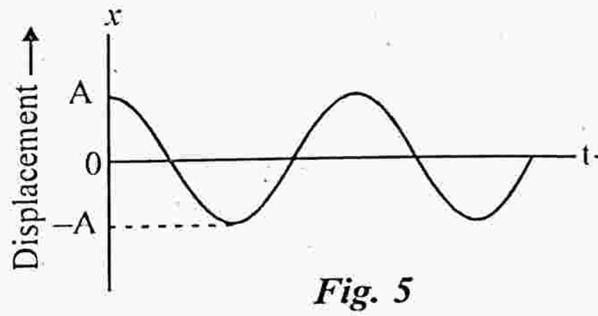


Fig. 5

The type of motion represented in fig.(4) is called SHM (Simple Harmonic Motion). It is a periodic motion expressed by a harmonic function of time. The quantity 'A' is called the amplitude of the motion. Maximum and minimum values of $\cos\omega t$ are ± 1 and corresponding $x(t)$ has values $\pm A$ (at extreme positions). For two arbitrary amplitudes, the graph looks like fig. 7(a). The quantity $\overline{\omega t + \phi}$ is known as phase of the motion. It gives the state or conditions of motion at a given instant. 'ϕ' is called phase constant (or phase angle or epoch at t = 0). The value of ϕ depends on the displacement and velocity of the particle at t = 0. The variations of the periodic motion with different ϕ values typical, are shown in fig. 7(b). The angular frequency $\omega = \frac{2\pi}{T}$ and its unit is rad s^{-1}.

$$\underset{\text{Displacement}}{x(t)} = \underset{\text{Amplitude}}{A}\ \cos(\overbrace{\underset{\text{Angular frequency}}{\omega t} + \underset{\text{Phase constant}}{\phi}}^{\text{Phase}})$$

Fig. 6

Fig. 8 illustrates the significance of period T.

In this figure, SHM represented by curve 'a' has a period T and that

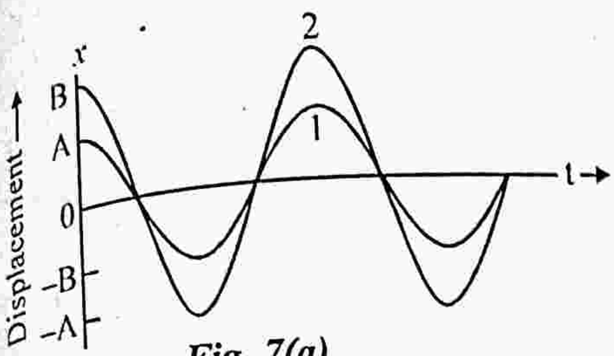


Fig. 7(a)

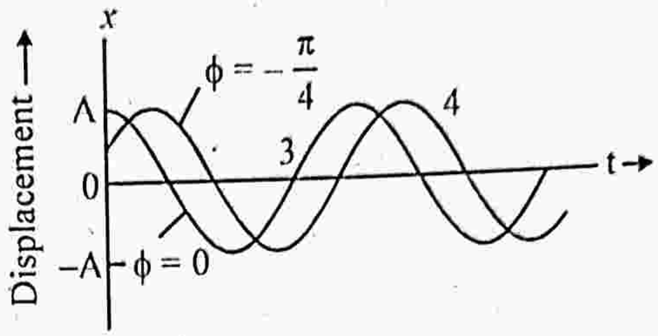


Fig. 7(b)

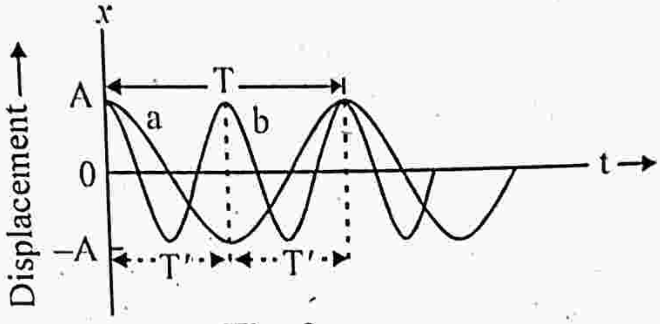


Fig. 8

represented by curve 'b' has a period $T' = \frac{T}{2}$.

Solved Examples

4. Which of the following functions of time represents (a) simple harmonic motion and (b) periodic but not simple harmonic? Give the period for each case.

i. $\sin\omega t - \cos\omega t$ ii. $\sin^2\omega t$

Sol.

a. $\sin\omega t - \cos\omega t$

$$= \sin\omega t - \sin\left(\frac{\pi}{2} - \omega t\right)$$

$$= 2\cos\left(\frac{\pi}{4}\right)\sin\left(\omega t - \frac{\pi}{4}\right)$$

$$= \sqrt{2}\sin\left(\omega t - \frac{\pi}{4}\right)$$

This function represents a simple harmonic motion having a period $T = \frac{2\pi}{\omega}$ and a phase angle $\left(-\frac{\pi}{4}\right)$ or $\left(\frac{7\pi}{4}\right)$

b. $\sin^2\omega t$

$$= \frac{1}{2} - \frac{1}{2}\cos 2\omega t$$

The function is periodic having a period $T = \frac{\pi}{\omega}$. It also represents a harmonic motion with the point of equilibrium occurring at $\frac{1}{2}$ instead of zero.

14.4 Simple Harmonic Motion and Uniform Circular Motion

Galileo in 1610 studied the motion of the moons of Jupiter. According to him, the motion is back and forth about Jupiter as origin (This means SHM). We now know that Callisto, one of Jupiter's moons move in circle with uniform speed. This conclues that Galileo observed the projected motion of the moons of Jupiter.

Fig. 9 shows the uniform circular motion of a particle 'P' with angular velocity 'ω'. The circle of radius (A say) is called reference circle. At any instant the position is projected on the X- axis. P' is the position. From figure, $x(t) = A\cos\overline{\omega t + \phi}$. The motion of P' on X-axis is synchronized with the motion of P on the reference circle. Hence, ***SHM is the projection of uniform circular motion on the diameter (actually on any diameter) of the reference circle.***

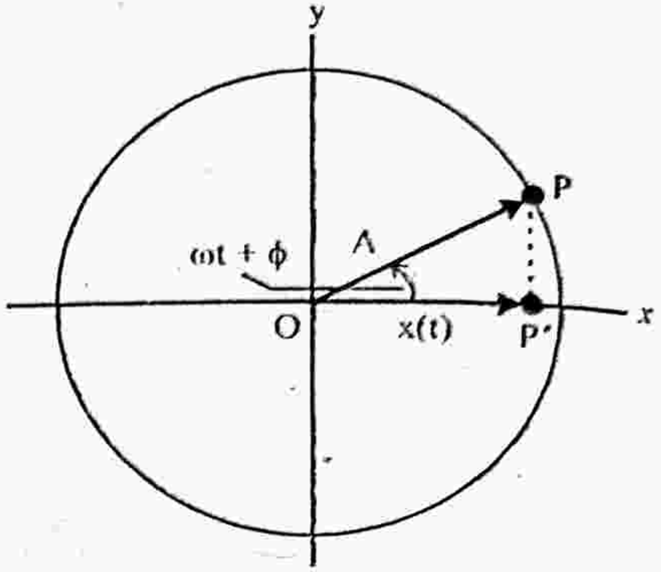
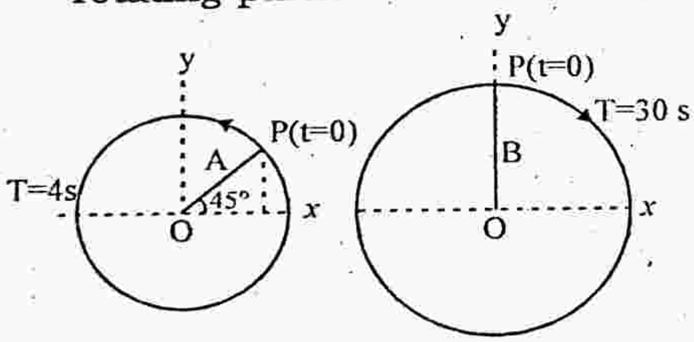


Fig. 9

Solved Examples

5. The figure depicts two circular motions. The radius of the circle, the period of revolution, the initial position and the sense of revolution are indicated on the figures. Obtain the simple harmonic motions of the x-projection of the radius vector of the rotating particle P in each case.



Sol.

a. At t = 0, OP makes an angle of 45° $= \frac{\pi}{4}$ rad with the (positive direction of) x-axis. After time t, it covers an angle $\frac{2\pi}{T}t$ in the anticlockwise sense, and makes an angle of $\frac{2\pi}{T}t + \frac{\pi}{4}$ with the x-axis.

The projection of OP on the x-axis at time t is given by

$$x(t) = A\cos\left(\frac{2\pi}{T}t + \frac{\pi}{4}\right)$$

For T = 4s

$x(t) = A\cos\left(\frac{2\pi}{4}t + \frac{\pi}{4}\right)$ which is a SHM of amplitude A, period 4s, and an initial phase $= \frac{\pi}{4}$.

b. In this case at t = 0, OP makes an angle of 90° $= \frac{\pi}{2}$ with the x-axis. After a time t, it covers an angle of $\frac{2\pi}{T}t$ in the clockwise sense and makes an angle of $\left(\frac{\pi}{2} - \frac{2\pi}{T}t\right)$ with the x-axis. The projection of OP on the x-axis at time t is given by

$$x(t) = B\cos\left(\frac{\pi}{2} - \frac{2\pi}{T}t\right)$$

$$= B\sin\left(\frac{2\pi}{T}t\right)$$

For T = 30s, $x(t) = B\sin\left(\frac{\pi}{15}t\right)$

Writing this as $x(t) = B\cos\left(\frac{\pi}{15}t - \frac{\pi}{2}\right)$ and comparing with eq. (1), we find that this represents a SHM of amplitude B, period 30 s and an initial phase of $-\frac{\pi}{2}$.

14.5 Velocity and Acceleration in Simple Harmonic Motion

The function $x(t) = A\cos\omega t$ is continuous and finite at any time (t). Therefore the velocity, acceleration and force are variable in nature; that also in magnitude and direction within periodic time itself.

Velocity at any time 't' is

$$v(t) = \frac{d}{dt}x(t) = -A\omega\sin\overline{\omega t + \phi} \quad \text{......... (2)}$$

$$= -A\omega\sqrt{1-\cos^2\overline{\omega t+\phi}}$$

$$= -A\omega\sqrt{1-\frac{x^2}{A^2}} = -\omega\sqrt{A^2 - x^2}$$

or $v = \omega\sqrt{A^2 - x^2}$ (numerically)

When $x = 0$, velocity is maximum, $v_{max} = A\omega$. This is at mean position. When $x = A$, i.e., at extreme position velocity is minimum and it is $v_{min} = 0$.

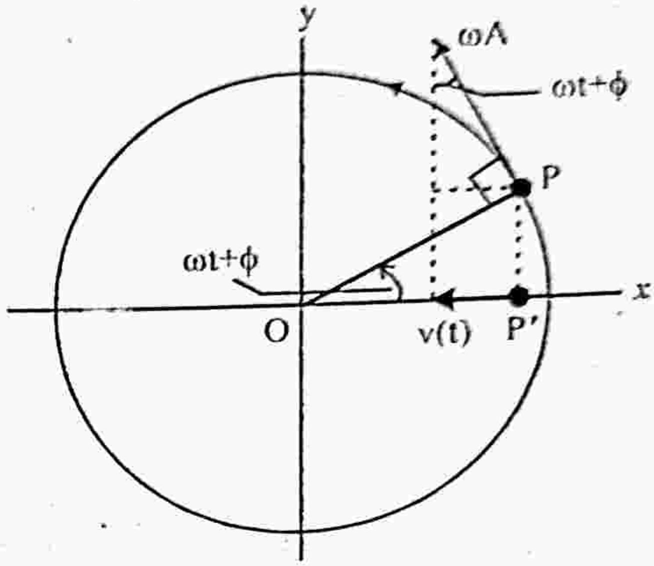


Fig. 10

This figure (Fig. 10), compared with fig. 9, shows the velocity of P, shown by the tangent through P, its value is $v = \omega A$. The projected value is $v(t) = -\omega A\sin\overline{\omega t + \phi}$ .... (3) as is seen from figure. The –ve sign is due to the component directed towards left, in the negative direction of x-axis. (3) is the instantaneous velocity.

Acceleration

Acceleration being derivative of velocity, acceleration $a(t) = \frac{d}{dt}v(t)$ ...... (4)

$$= \frac{d}{dt} - A\omega\sin\overline{\omega t+\phi} =$$

$$-A\omega^2\cos\overline{\omega t+\phi} = -\omega^2 x$$

This acceleration expression shows that

i. acceleration ∝ displacement

ii. always against displacement. The two conditions of SHM.

The projection of acceleration of P on x-axis is $a(t) = -\omega^2 A\cos\overline{\omega t+\phi}$

$= -\omega^2 x(t)$

At extreme position, acceleration is maximum $a_{max} = \pm\omega^2 A$

$\because x(t) = \mp A$ and at mean position

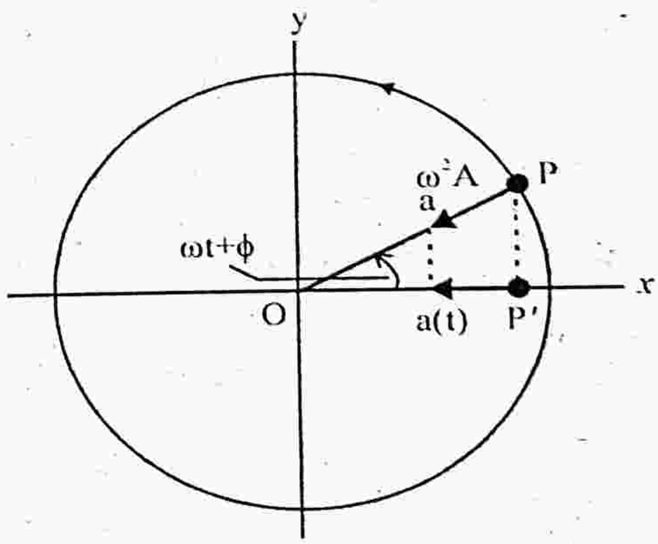


Fig. 11

it is $a_{min} = 0, \because x(t) = 0$

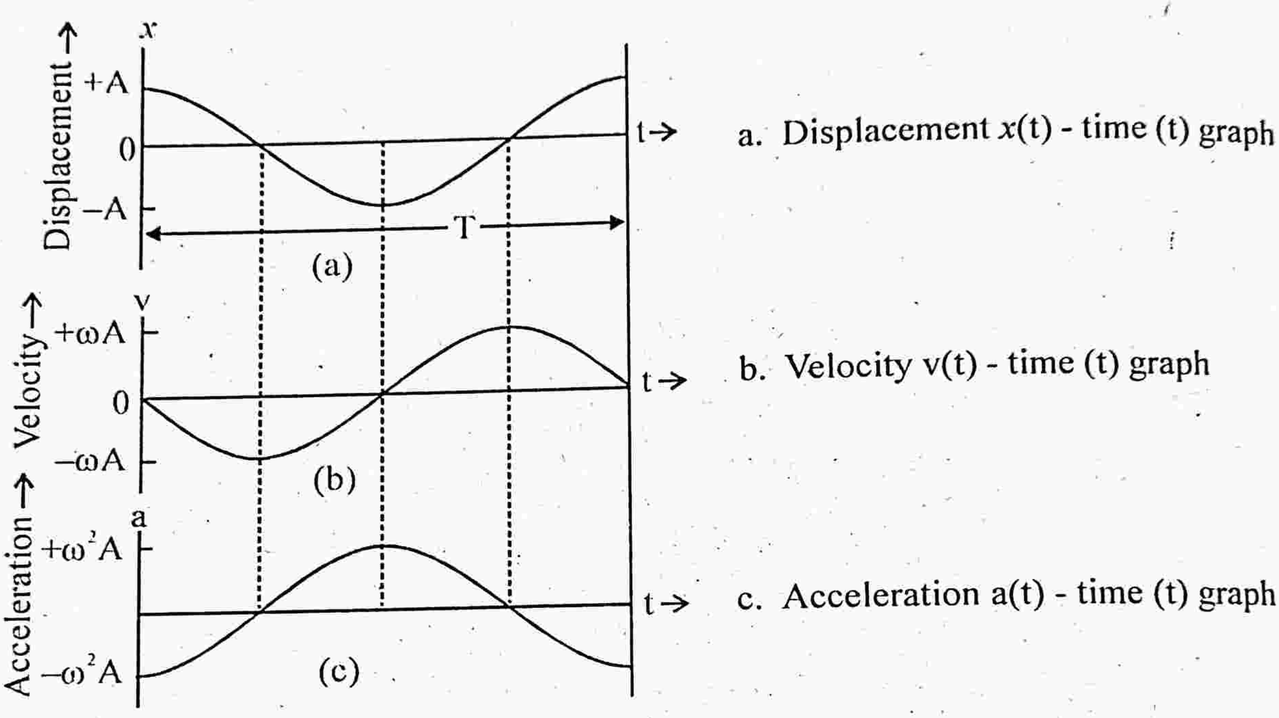


Fig. 12

Time period (T)

Time period *is defined as the time taken by the particle executing SHM to complete one vibration.*

Acceleration, $a = \omega^2 x$ (neglecting the negative sign)

$$\omega = \sqrt{\frac{a}{x}}$$

Time period, $T = \frac{2\pi}{\omega} = 2\pi \times \frac{1}{\sqrt{\left(\frac{a}{x}\right)}} = 2\pi\sqrt{\frac{1}{\left(\frac{a}{x}\right)}}$

$$T = 2\pi\sqrt{\frac{1}{\text{acceleration/unit displacement}}}$$

Frequency, $\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$