14.9 Damped Simple Harmonic Motion

A real simple pendulum executes oscillations whose amplitude decreases gradually. It is said to be a damped harmonic oscillator. See the fig. 16.

The vane is immersed in a liquid. During the oscillation of the spring, the amplitude decreases showing the decrease in mechanical energy. Usually damping is proportional to velocity. Hence, damping force $F_d = -bv$ where b is called damping constant and it depends on the characteristics of the liquid and the vane. The negative sign shows that F_d is against velocity. If the restor-

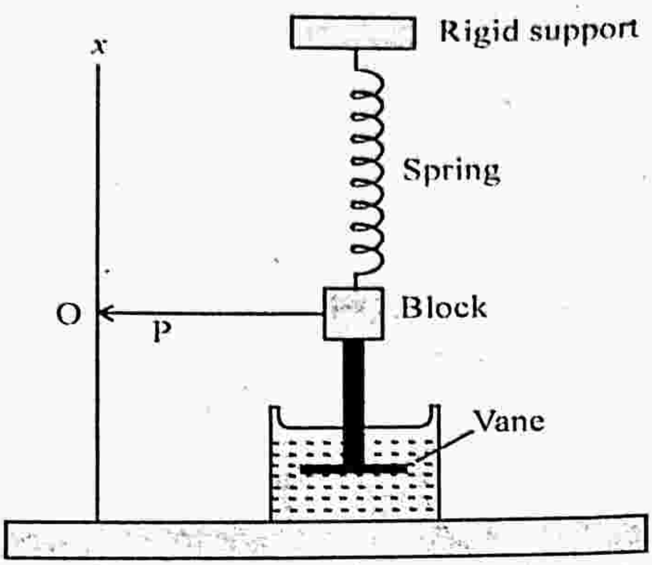


Fig. 16

Example for damped oscillator. One vane makes damped motion in the liquid

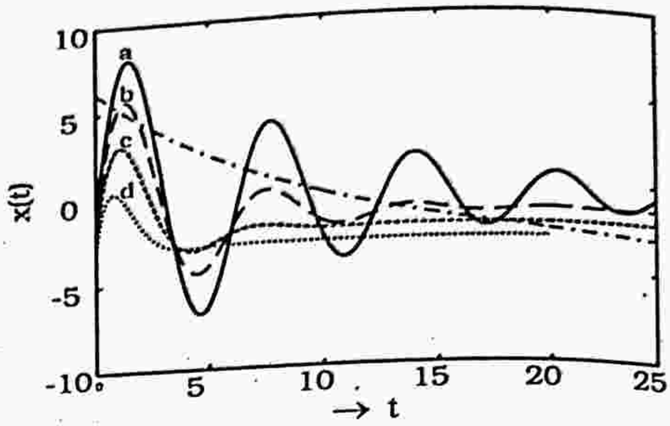


Fig. 17

Graphs show the variation of displacement with time. The damping force increases for curves a to d in progression

ing force is linear (due to spring) as $F_S = -kx$ where x is the displacement of the mass from equilibrium position. Then unbalanced force F is $F = -kx - bv$.

By Newton's second law $F_{(t)} = ma(t)$

$\therefore\ ma(t) = -k\,x(t) - bv(t)$ or $m.\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + kx = 0$. The solution is

$x(t) = Ae^{-\frac{bt}{2m}}.\cos(\omega' t + \phi)$ where $Ae^{-\frac{bt}{2m}}$ is the amplitude and ω' is the angular frequency.

$\omega' = \sqrt{\dfrac{k}{m} - \left(\dfrac{b}{2m}\right)^2}$ and is less than ω of the undamped oscillator. The motion is controlled by two functions (i) the exponential one $e^{-bt/2m}$, as a result the amplitude decreases fast and (ii) the harmonic term $\cos(\omega' t + \phi)$. The motion is approximately periodic.

When b = 0, the damping is zero and the oscillator becomes ideal again.

The total energy is $E(t) = \dfrac{1}{2}kA^2e^{-bt/m}$

The total energy decreases exponentially with time. Small damping means

$\dfrac{b}{\sqrt{km}}$ - the dimensionless ratio is much less than 1.

14.10 Forced Oscillations and Resonance

The reduction in the amplitude of a damped harmonic oscillator can be rectified by the application of a periodic applied force. Oscillation now becomes sustained. Let $F(t) = F_0 \cos\omega_d t$ be the periodic force applied. Considering the damping force and restoring force, the equation of motion becomes $m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\cos\omega_d t$.

The natural angular frequency ω and driving angular frequency ω_d influence the motion. But as the external force controls the motion, ω_d becomes effective. The displacement is

$x(t)=A\cos\overline{\omega_d t+\phi}$ after 't' seconds from the application of F(t). The amplitude is given by $A=\frac{F_0}{\left[m^2(\omega^2-\omega_d^2)^2+b^2\omega_d^2\right]^{1/2}}$ and $\tan\phi=\frac{-v_0}{\omega_d x_0}$, where v_0 and x_0 are initial velocity and displacement when t = 0.

Small damping

In this case, the second term is negligible and therefore $A=\frac{F_0}{m(\omega^2-\omega_d^2)}$. It is controlled by ω and ω_d.

Resonance

When $\omega_d\simeq\omega$, $m(\omega^2-\omega_d^2)$ would be much less than $b\omega_d$, then amplitude becomes $A=\frac{F_0}{b\omega_d}$. It is clear that the maximum possible amplitude for a given driving frequency is controlled by ω_d and the damping force and is never infinity.

With $\omega=\omega_d$, amplitude becomes a maximum and this phenomenon is called resonance.

Let us see the influence of the oscillations on other oscillator's coupled.

A system of five simple pendulums are suspended from a common rope. Pendulums 1 and 4 are of same length, others are different. When 1 is set into motion, all of them oscillate in due course but out of phase except 4 which oscillates in phase. The condition of resonance is satisfied only in 4. It is interesting to note that in an earthquake short and tall structures remain unaffected while the medium height structures fall down. This is because the frequency of seismic wave agrees with that of medium structures.

A military platoon is instructed to break the steps while passing through a bridge. If the men in platoon move in steps, the corresponding frequency may coincide with the natural frequency of the bridge which may bring the bridge to resonance and ultimately a collapse.

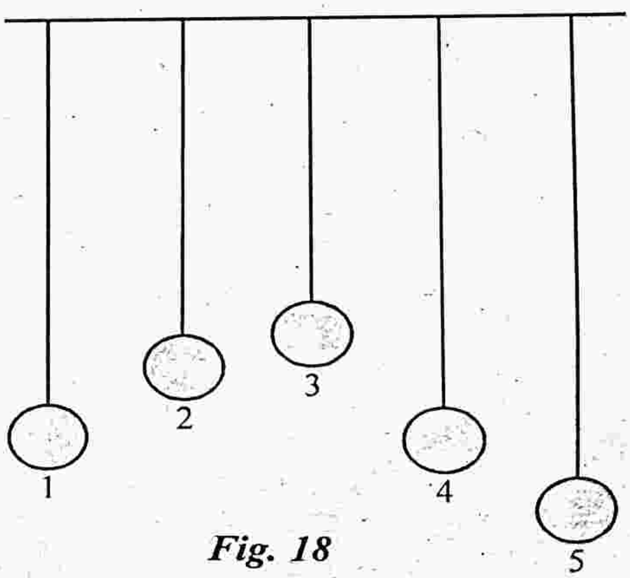


Fig. 18

Solutions for NCERT Exercises

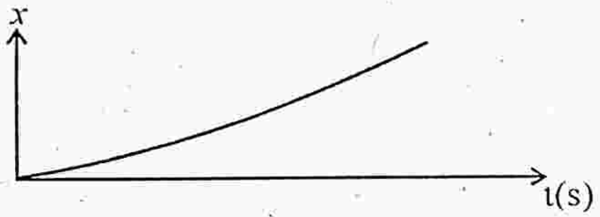
1. Which of the following examples represent periodic motion?
   a. A swimmer completing one (return) trip from one bank of a river to the other and back.
   b. A freely suspended bar magnet displaced from its N-S direction and released.
   c. A hydrogen molecule rotating about its center of mass.
   d. An arrow released from a bow.

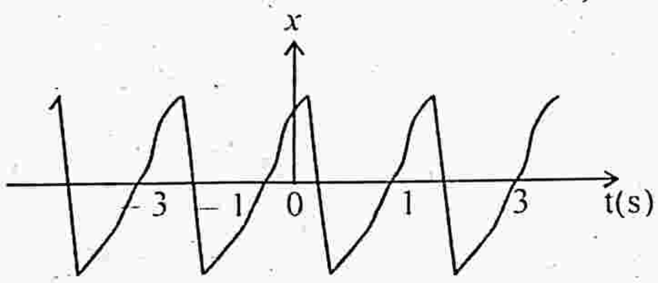
Ans. (b) and (c) are periodic motions. (a) and (d) not.

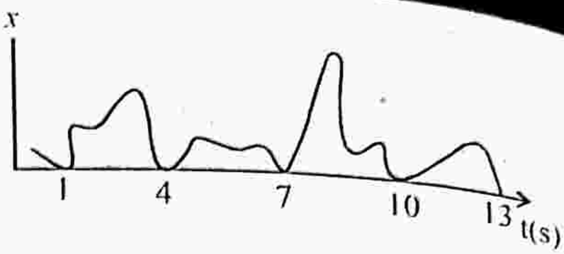
2. Which of the following examples represents (nearly) simple harmonic motion and which represents periodic but not simple harmonic motion?
   a. The rotation of earth about its axis.
   b. Motion of an oscillating mercury column in a U-tube.
   c. Motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point.
   d. General vibrations of a polyatomic molecule about its equilibrium position.

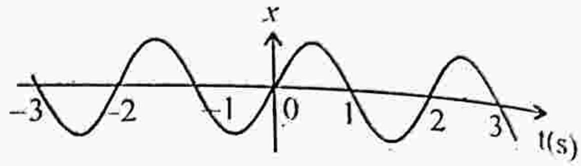
Ans. (b) and (c) SHM. (a) and (d) periodic but not SHM.

3. The figure depicts four *x-t* plots for linear motion of a particle. Which of the plots represent periodic motion? What is the period of motion (in case of periodic motion)?

a.


b.


c.


d.


Ans. (b) and (d) periodic

4. Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion? Give period for each case of periodic motion (ω is any positive constant)

   a. $\sin\omega t - \cos\omega t$ b. $\sin^3\omega t$

   c. $3\cos\left(\frac{\pi}{4} - 2\omega t\right)$

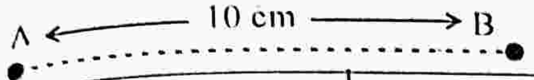
   d. $\cos\omega t + \cos 3\omega t + \cos 5\omega t$

   e. $\exp(-\omega^2 t^2)$ f. $1 + \omega t + \omega^2 t^2$

Ans. (a) and (c) $\rightarrow$ SHM, (b) and (d) periodic but not SHM, (e) and (f) non periodic

5. A particle is in linear simple harmonic motion between two points, A and B, 10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is
   a. at the end A
   b. at the end B
   c. at the mid-point of AB going towards A
   d. at 2 cm away from B going towards A
   e. at 3 cm away from A going towards B and
   f. at 4 cm away from B going towards A.

Ans.



	Acceleration	Velocity	Force
a.	$+5\omega^2(+)$	0	$+5m\omega^2(+)$
b.	–	0	–
c.	0	-5ω	0
d.	–	–	–
e.	+	+	+
f.	+	–	+

6. Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion?
a. $a = 0.7x$ b. $a = -200x^2$
c. $a = -10x$ d. $a = 100x^3$

Ans. c

7. The motion of a particle executing simple harmonic motion is described by the displacement function, $x(t) = A\cos(\omega t + \phi)$
If the initial (t = 0) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle? The angular frequency of the particle is π s^{-1}. If instead of the cosine function, we choose the sine function to describe the SHM : $x = B\sin(\omega t+\alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions.

Ans. $x(t) = A\cos(\omega t + \phi)$. When t = 0,

x = 1 cm, $v_m = \omega$ cm/s

A = ?, ϕ = ? $\omega = \pi s^{-1}$.

When t = 0, $x(0) = A\cos\phi$

$$v = -A\omega\sin\overline{\omega t+\phi}$$

$$1 = A\cos\phi \quad \text{..................} \quad (1)$$

$$\omega = -A\omega\sin\phi$$

$$1 = -A\sin\phi \quad \text{............} \quad (2)$$

Squaring $1 = A^2\cos^2\phi$ ∴ $A^2 = 2$

$A = \pm\sqrt{2}$

$-\tan\phi = 1$, $\tan\phi = -1$ ∴ $\phi = \frac{3\pi}{4}$

If we take $x = B\sin(\omega t + \alpha)$, x = 1 cm, t = 0, $v_{ini} = \omega\ cms^{-1}$

$x = B\sin\overline{\omega t + \alpha}$, t = 0, becomes

$x_0 = B\sin\alpha$

i.e., $1 = B\sin\alpha$ .... (1) and

$v = B\omega\cos\overline{\omega t + \alpha}$

$1 = B\cos\alpha$ .... (2) $\omega = B\cos\overline{\omega t + \alpha}.\omega$

$B = \sqrt{2}$ $\tan\alpha = 1$, $\alpha = \frac{\pi}{4}$ or $\alpha = \frac{5\pi}{4}$

8. A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body?

Ans. 20 cm corresponds to 50 kg
i.e., kx = 50 kg wt = 50 × 9.8 N,
x = 20 cm = 20 × 10^{-2} m

$$k = \frac{50\times 9.8}{20\times 10^{-2}} = 2.5 \times 9.8 \times 100$$

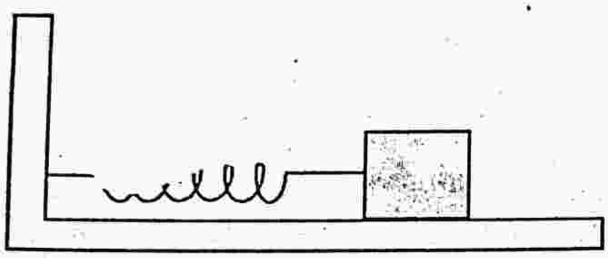
$$= 25 \times 98 \text{ N/m}$$

$$T = 2\pi\sqrt{\frac{m}{k}}, \qquad T^2 = 4\pi^2 \times \frac{m}{k}$$

$\therefore m = \frac{kT^2}{4\pi^2}$

$= \frac{25 \times 98 \times 0.36}{4 \times (3.14)^2} = 22.3$ kg

9. A spring having a spring constant 1200 N m^{-1} is mounted on a horizontal table as shown in figure. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.



Determine (i) the frequency of oscillations, (ii) maximum acceleration of the mass, and (iii) the maximum speed of the mass.

Ans.

k = 1200 Nm^{-1} m = 3 kg,

x = 2.0 cm

ν = ?, a_{max} = ?, v_{max} = ?

f = kx = 1200 × 2 × 10^{-2} = 24 N

i. $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2 \times 3.14}\sqrt{\frac{1200}{3}}$

$= \frac{2 \times 10}{2 \times 3.14} = \frac{10}{\pi} s^{-1}$

ii. $a = \frac{f}{m} = \frac{24}{3} = 8 \ ms^{-2}$,

iii. $v_{max} = 0.4 \ ms^{-1}$

10. In the above problem, let us take the position of mass when the spring is unstreched as x = 0, and the direction from left to right as the positive direction of x-axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch (t = 0), the mass is

a. at the mean position,

b. at the maximum stretched position, and

c. at the maximum compressed position.

In what way do these functions for SHM differ from each other, in frequency, in amplitude or the initial phase?

Ans.

a. x = 2 sin 20t

b. x = 2 cos 20 t

c. x = –2 cos 20 t

Only in initial phase.

11. The figures correspond to two circular motions. The radius of the circle, the period of revolution, the initial position, and the sense of revolution (i.e. clockwise or anti-clockwise) are indicated on each figure.

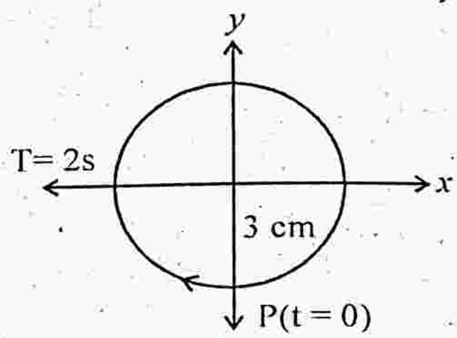


Fig. (a)

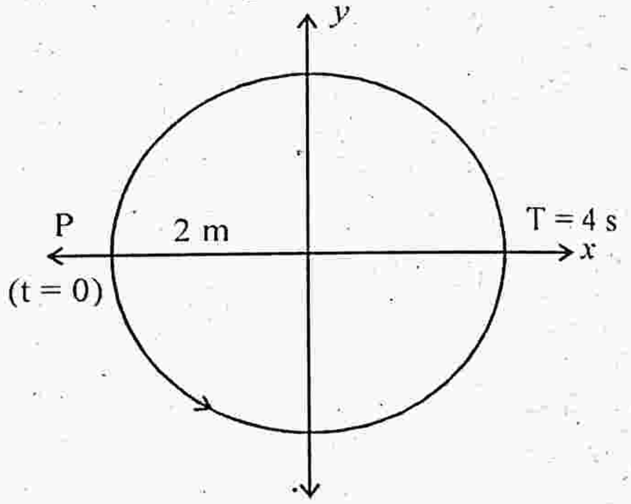
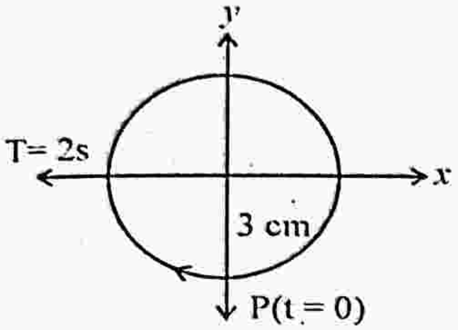
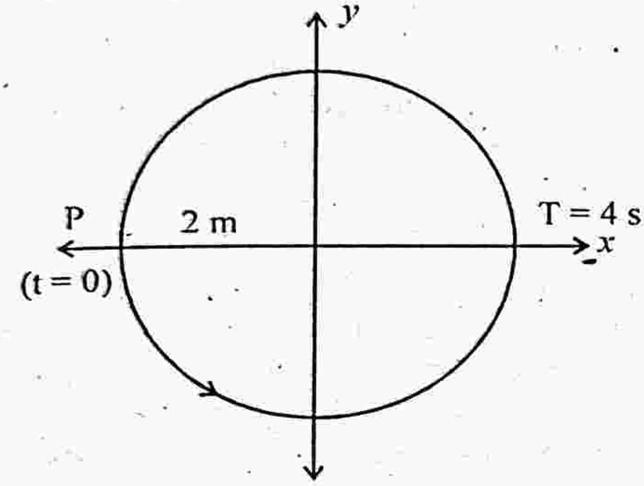


Fig. (b)

Obtain the corresponding simple harmonic motions of the x-projection of the radius vector of the revolving particle P, in each case.

Ans.





a. $x = 3\cos\left(\omega t + \frac{\pi}{2}\right) = -3\sin \pi t$

b. $x = 2\cos(\omega t + \pi) = -2\cos\frac{\pi}{2}t$

12. Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial (t = 0) position of the particle, the radius of the circle, and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case: (x is in cm and t is in s).

a. $x = -2\sin\left(3t + \frac{\pi}{3}\right)$

b. $x = \cos\left(\frac{\pi}{6 - t}\right)$

c. $x = 3\sin\left(2\pi t + \frac{\pi}{4}\right)$

d. $x = 2\cos \pi t$

Ans.

Express in the form $x = A\cos(\omega t + \phi)$, ϕ is the angle, the initial radius vector of the particle makes with the positive direction of x-axis.

13. The figure (a) shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. A force F applied at the free end stretches the spring. Figure (b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Fig. (b) is stretched by the same force F.

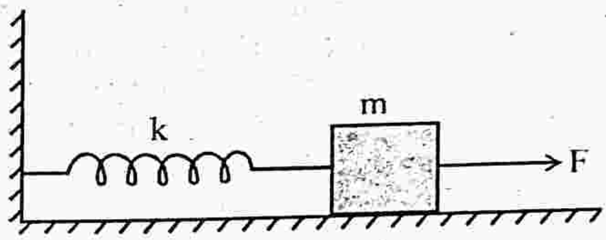


Fig. (a)

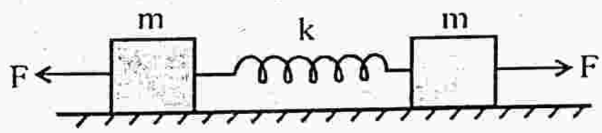


Fig. (b)

a. What is the maximum extension of the spring in the two cases ?

b. If the mass in Fig. (a) and the two masses in Fig. (b) are released, what is the period of oscillation in each case?

Ans.

a. Extension in both cases is $x = \frac{F}{k}$

b. In the first case F = ma = kx

$\therefore \frac{a}{x} = \frac{k}{m}$ and $T = 2\pi\sqrt{\frac{m}{k}}$

In the second case, F = ma

$= k'x' \qquad \therefore \frac{a}{x'} = \frac{k'}{m}$.

But $k' \times \frac{l}{2} = kl = k' = 2k$

$\therefore \frac{a}{x'} = \frac{2k}{m}$ and $T = 2\pi\sqrt{\frac{m}{2k}}$

14. The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min, what is its maximum speed?

Ans. A = 0.5 m ω = 200 rad/min

$v_{max} = A\omega = 0.5 \times 200$

= 100 m/min

15. The acceleration due to gravity on the surface of moon is 1.7 ms^{-2}. What is the time period of a simple pendulum on the surface of moon if its time period on the surface of earth is 3.5 s? (g on the surface of earth is 9.8 ms^{-2})

Sol.

Let T_E - period on Earth

T_M - period on Moon

g_E - acceleration due to gravity on Earth

g_M - acceleration due to gravity on Moon, then

$$T = 2\pi\sqrt{\frac{l}{g}}, \quad \frac{T_M^2}{T_E^2} = \frac{g_E}{g_H}$$

$$\therefore T_M = \sqrt{\frac{g_E}{g_H}}.T_E = \sqrt{\frac{9.8}{1.7}} \times 3.5 = 8.4 \text{ s}$$

16. Answer the following questions.

a. Time period of a particle in SHM depends on the force constant k and mass m of the particle: $T = 2\pi\sqrt{\frac{m}{k}}$. A simple pendulum executes SHM approximately. Why then is the time period of a pendulum independent of the mass of the pendulum?

b. The motion of a simple pendulum is approximately simple harmonic for small angle oscillations. For larger angles of oscillation, a more involved analysis shows that T is greater than $2\pi\sqrt{\frac{l}{g}}$. Think of a qualitative argument to appreciate this result.

c. A man with a wrist watch on his hand falls from the top of a tower. Does the watch give correct time during the free fall?

d. What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity?

Sol.

a. k determines the restoring force. As m increases $mg\sin\theta$ also increases. i.e., $k \propto m$. Hence in simple pendulum m gets cancelled.

b. Instead of $\sin\theta$, if we put θ, which is small. i.e., $\theta < \sin\theta$, the restoring force $mg\sin\theta$ decreases which will increase the period.

c. Yes   d. Zero

17. A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period ?

Sol. $$T = 2\pi\sqrt{\frac{l}{\sqrt{\left(\frac{v^2}{R}\right)^2 + (g)^2}}}.$$

$\because$ Net acceleration is $a = \sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}$