2. RELATIONS AND FUNCTIONS

Relations

A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product A×B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A ×B. The second element is called the *image* of the first element

The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the *domain* of the relation R.

The set of all second elements in a relation R from a set A to a set B is called the *range* of the relation R. The whole set B is called the *codomain* of the relation R. Note that range ⊂ codomain.

Remarks

i. A *relation* may be represented algebraically either by the *Roster method* or by the *Set-builder method*.

ii. An arrow diagram is a visual representation of a relation

Example 7 Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by

R = {(x, y) : $y = x + 1$ }

(i) Depict this relation using an arrow diagram.

(ii) Write down the domain, codomain and range of R.

Solution (i) By the definition of the relation,

R = {(1,2), (2,3), (3,4), (4,5), (5,6)}.

The corresponding arrow diagram is

(ii) We can see that the domain ={1, 2, 3, 4, 5,}

Similarly, the range = {2, 3, 4, 5, 6}

and the codomain = {1, 2, 3, 4, 5, 6}.

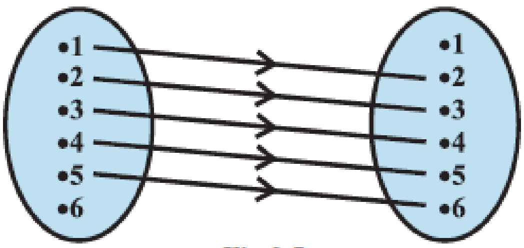


Fig 2.5

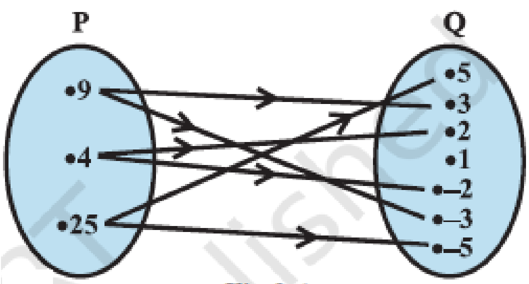


Fig 2.6

Example 8 The Fig 2.6 shows a relation between the sets P and Q. Write this relation (i) in set-builder form, (ii) in roster form. What is its domain and range?

Solution It is obvious that the relation R is "x is the square of y".

(i) In set-builder form, R = {(x, y): x is the square of y, $x \in$ P, $y \in$ **Q**}

(ii) In roster form, R = {(9, 3), (9, –3), (4, 2), (4, –2), (25, 5), (25, –5)}
The domain of this relation is {4, 9, 25}.
The range of this relation is {– 2, 2, –3, 3, –5, 5}.
Note that the element 1 is not related to any element in set P.
The set Q is the codomain of this relation.
NOTE: The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A ×B. If $n(A) = p$ and $n(B) = q$, then
$n(A \times B) = pq$ and the total number of relations is 2^{pq}.

Example 9 Let A = {1, 2} and B = {3, 4}. Find the number of relations from A to B.
Solution We have,
A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}. Since $n(A\times B) = 4$, the number of subsets of A×B is 24. Therefore, the number of
relations from A into B will be 24.
Remark A relation R from A to A is also stated as a relation on A.
Example 10
Let A = {1, 2, 3,...,14}. Define a relation R from A to A by
R = {(x, y) : $3x – y = 0$, where $x, y \in$ A}. Write down its domain, codomain and range.
Solution
The relation R from A to A is given as R = {(x, y): $3x – y = 0$, where $x, y \in$ A}
i.e., R = {(x, y): $3x = y$, where $x, y \in$ A}
∴ R = {(1, 3), (2, 6), (3, 9), (4, 12)}
The domain of R is the set of all first elements of the ordered pairs in the relation.
∴ Domain of R = {1, 2, 3, 4}
The whole set A is the codomain of the relation R.
∴ Codomain of R = A = {1, 2, 3… 14}
The range of R is the set of all second elements of the ordered pairs in the relation.
∴Range of R = {3, 6, 9, 12}
Example 11
A = {1, 2, 3, 5} and B = {4, 6, 9}. Define a relation R from A to B by
R = {(x, y): the difference between x and y is odd; $x \in$ A, $y \in$ B}. Write R in roster form.
Solution
A = {1, 2, 3, 5} and B = {4, 6, 9}
R = {(x, y): the difference between x and y is odd; $x \in$ A, $y \in$ B}
∴ R = {(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)}

Example 12 The Fig2.7 shows a relationship between the sets P and Q. Write this relation (i) in set-builder form (ii) roster form. What is its domain and range?

Solution

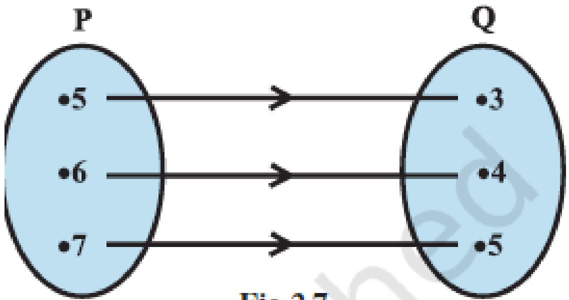


Fig 2.7

According to the given figure, P = {5, 6, 7}, Q = {3, 4, 5}

(i) R = {(x, y): $y = x - 2$; $x \in$ P} or R = {(x, y): $y = x - 2$ for x = 5, 6, 7}

(ii) R = {(5, 3), (6, 4), (7, 5)}

Domain of R = {5, 6, 7}

Range of R = {3, 4, 5}

Example 13 Let A = {1, 2, 3, 4, 6}. Let R be the relation on A defined by {(a, b): a , b $\in$A, b is exactly divisible by a}.

(i) Write R in roster form

(ii) Find the domain of R

(iii) Find the range of R.

Solution

A = {1, 2, 3, 4, 6}, R = {(a, b): a, b $\in$ A, b is exactly divisible by a}

(i) R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)}

(ii) Domain of R = {1, 2, 3, 4, 6}

(iii) Range of R = {1, 2, 3, 4, 6}

Example 14 Write the relation R = {(x, x3) : x is a prime number less than 10} in roster form

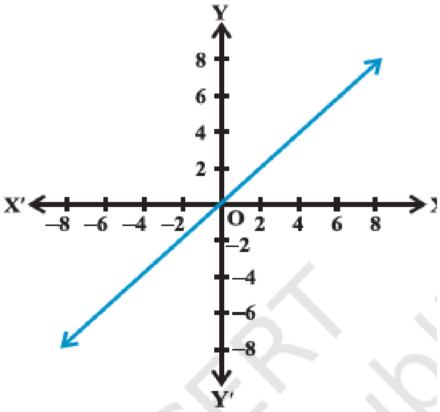
Solution

R = {(x, x3): x is a prime number less than 10} The prime numbers less than 10 are 2, 3, 5, and 7.

∴ R = {(2, 8), (3, 27), (5, 125), (7, 343)}

Some functions and their graphs

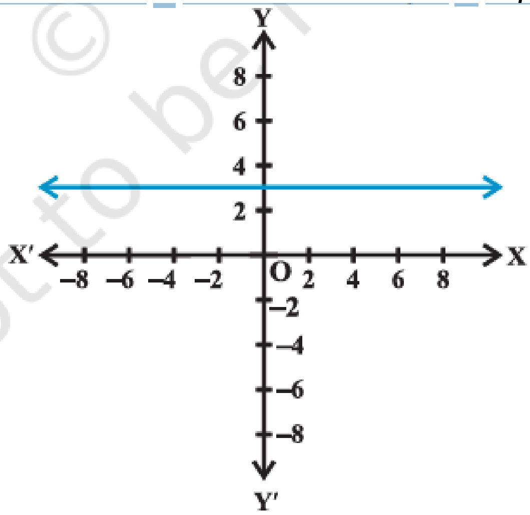
(i) **Identity function** Let **R** be the set of real numbers. Define the real valued function $f : \mathbf{R} \to \mathbf{R}$ by $y = f(x) = x$ for each $x \in \mathbf{R}$. Such a function is called the *identity function*. Here the domain and range of f are **R**. The graph is a straight line as shown in Fig 2.8. It passes through the origin.



$f(x) = x$

Fig 2.8

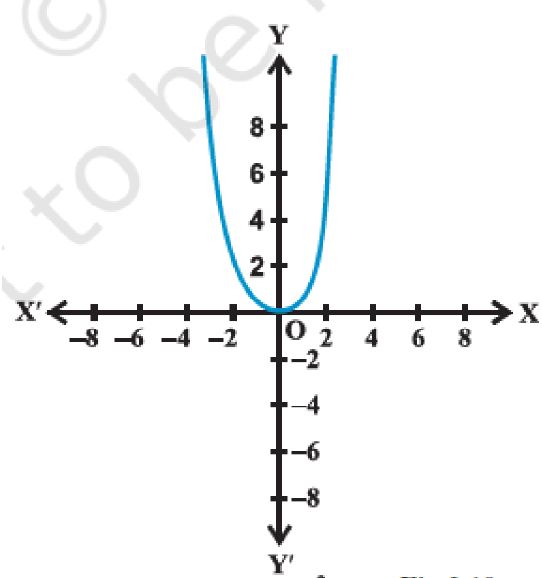
(ii) **Constant function** Define the function $f: \mathbf{R} \to \mathbf{R}$ by $y = f(x) = c, x \in \mathbf{R}$ where c is a constant and each $x \in \mathbf{R}$. Here domain of f is **R** and its range is $\{c\}$.



$f(x) = 3$

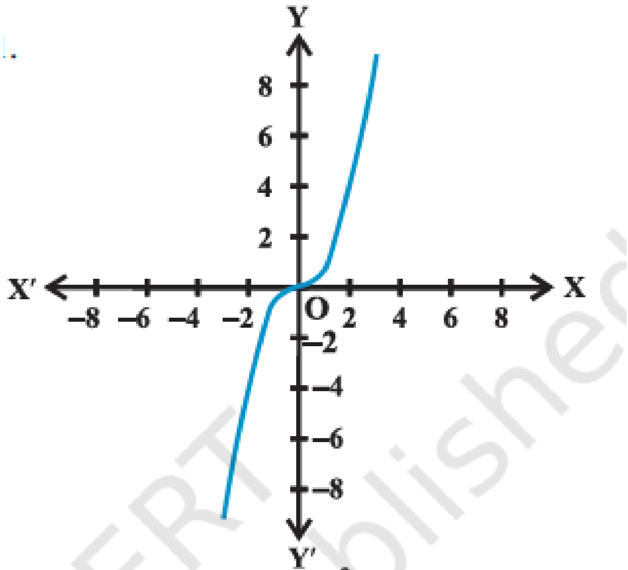
Fig 2.9

(iii) Polynomial function Define the function f: $\mathbf{R} \to \mathbf{R}$ by $y = f(x) = x^2$, $x \in \mathbf{R}$. Domain of $f = \{x : x \in \mathbf{R}\}$. Range of $f = \{x^2 : x \in \mathbf{R}\}$. The graph of f is given by Fig 2.10



$f(x) = x^2$ Fig 2.10

(iv) the graph of the function $\boldsymbol{f}$:$\mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$, $x \in \mathbf{R}$.



$f(x) = x^3$

Fig 2.11

Rational functions

Define the real valued function $f : \mathbf{R} - \{0\} \rightarrow \mathbf{R}$ defined by 1 $f(x) = \frac{1}{x}$, $x \in \mathbf{R} - \{0\}$. The domain is all real numbers except 0 and its range is also all real numbers except 0. The graph of f is given in Fig 2.12.

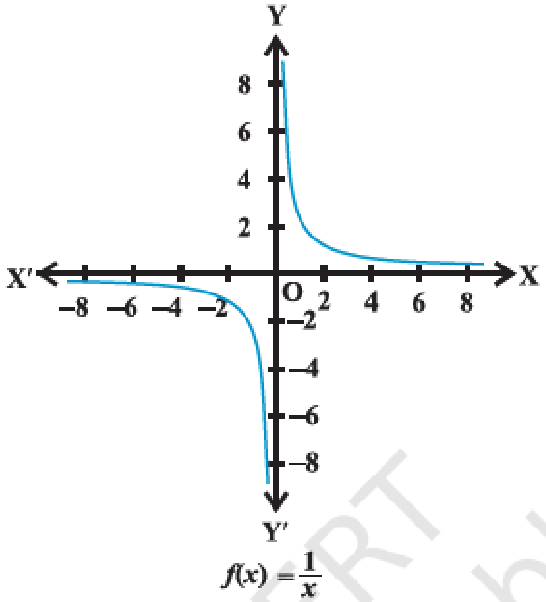


Fig 2.12

(v) **The Modulus function** The function f: **R**→**R** defined by $f(x) = |x|$ for each $x \in \mathbf{R}$ is called *modulus function*. For each non-negative value of x, $f(x)$ is equal to x. But for negative values of x, the value of $f(x)$ is the negative of the value of x, i.e., $f(x) = \begin{Bmatrix} x, x \geq 0 \\ -x, x < 0 \end{Bmatrix}$

The graph of the modulus function is given in Fig 2.13

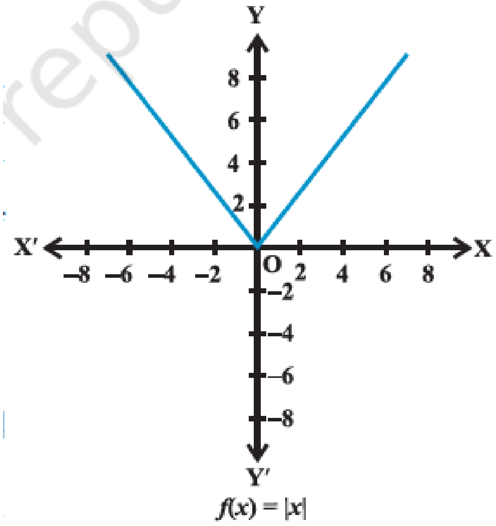
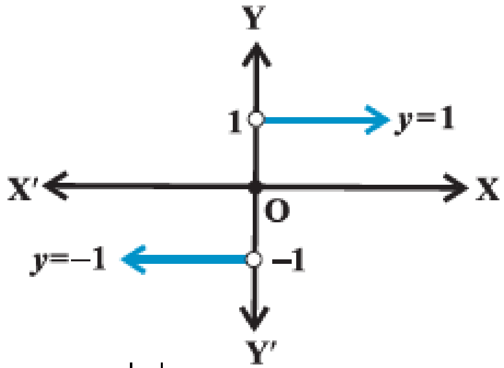


Fig 2.13

vi) **Signum function** The function *f*:**R**→**R** defined by

$$f(x) = \begin{Bmatrix} 1, & if\ x > 0 \\ 0, & if\ x = 0 \\ -1, & if\ x > 0 \end{Bmatrix}$$

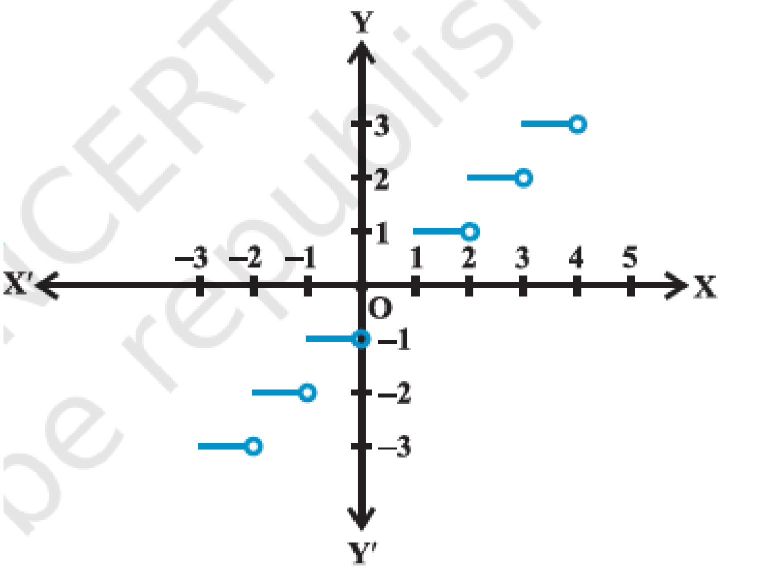
is called the *signum function*. The domain of the signum function is **R** and the range is the set {–1, 0, 1}. The graph of the signum function is given by the Fig 2.14.



$f(x) = \frac{|x|}{x}, x \neq 0$ and 0 for $x = 0$

Fig 2.14

(vii) **Greatest integer function**
The function *f*: **R** → **R** defined by *f*(*x*) = [*x*], *x* ∈**R** assumes the value of the greatest integer, less than or equal to *x*. Such a function is called the *greatest integer function*.
From the definition of [*x*], we can see that
[*x*] = –1 for –1 ≤ *x* < 0
[*x*] = 0 for 0 ≤ *x* < 1
[*x*] = 1 for 1 ≤ *x* < 2
[*x*] = 2 for 2 ≤ *x* < 3 an and so on.
The graph of the function is shown in Fig 2.15.



$f(x) = [x]$

Fig 2.15

Example 22 The function f is defined by

$$f(x) = \begin{cases} 1-x, & x<0 \\ 1, & x=0 \\ x+1, & x>0 \end{cases}$$

Draw the graph of $f(x)$.

Solution Here, $f(x) = 1 - x$, $x < 0$, this gives

$f(-4) = 1 - (-4) = 5$;

$f(-3) = 1 - (-3) = 4$,

$f(-2) = 1 - (-2) = 3$

$f(-1) = 1 - (-1) = 2$; etc,

and $f(1) = 2$, $f(2) = 3$, $f(3) = 4$

$f(4) = 5$ and so on for $f(x) = x + 1$, $x > 0$.

Thus, the graph of f is as shown in Fig 2.17

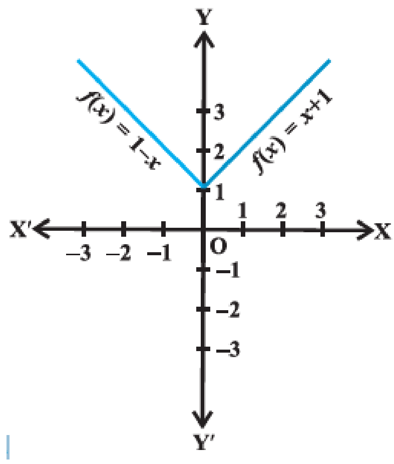
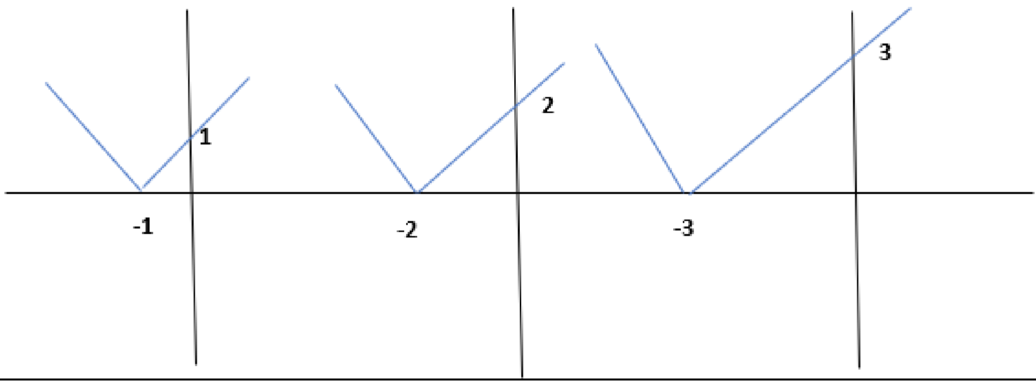


Fig 2.17

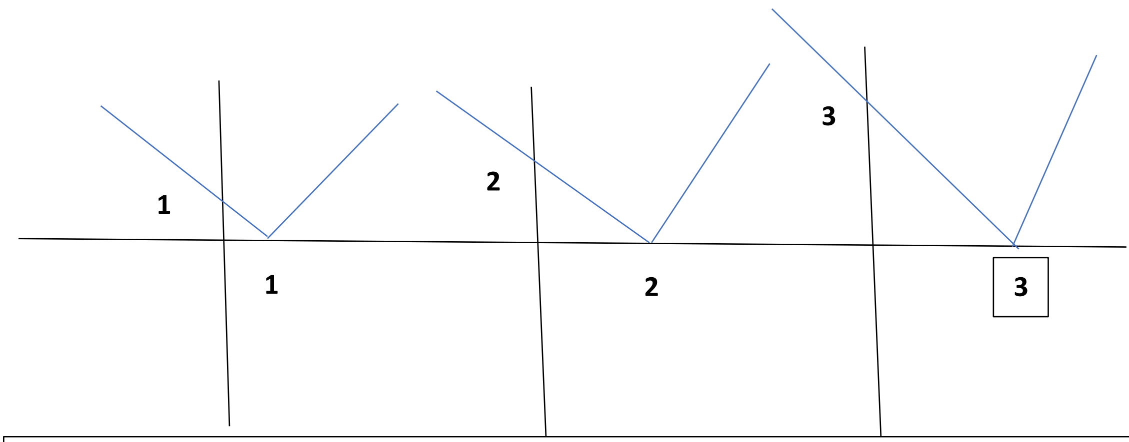
SOME MORE GRAPHS

Shift towards Left



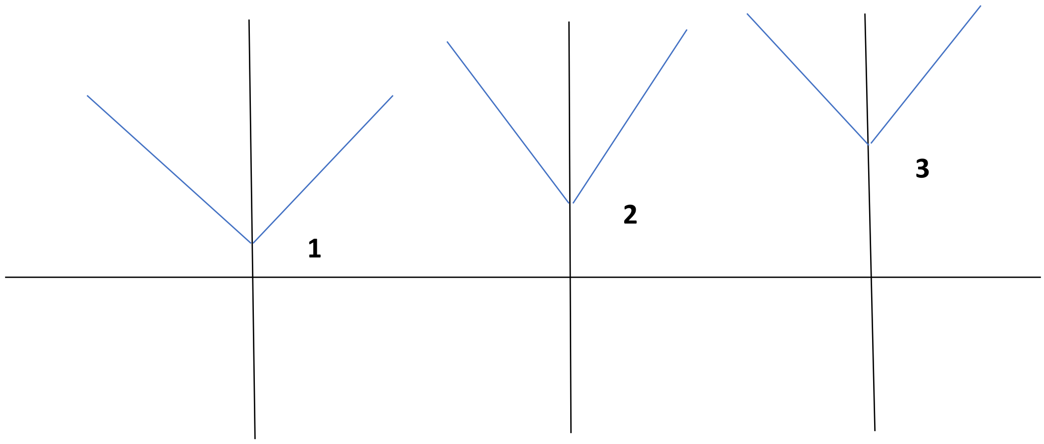
$f(x) = \lvert x+1 \rvert$	$f(x) = \lvert x+2 \rvert$	$f(x) = \lvert x+3 \rvert$

Shift towards Right



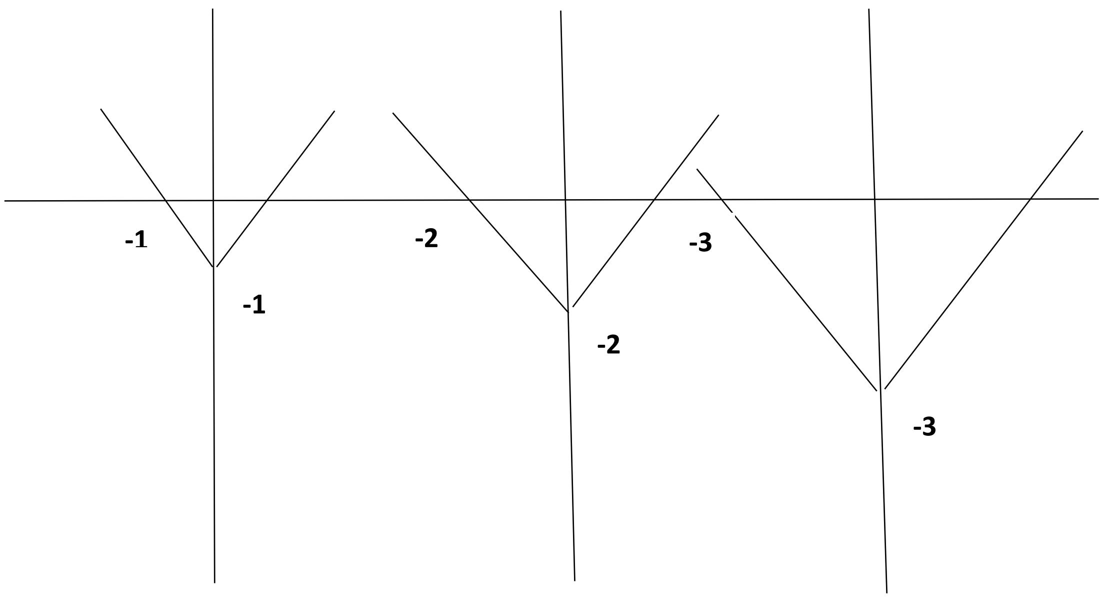
$f(x) = \lvert x-1 \rvert$	$f(x) = \lvert x-2 \rvert$	$f(x) = \lvert x-3 \rvert$

Shift towards +ve y- axis



$f(x) = \|x\| + 1$	$f(x) = \|x\| + 2$	$f(x) = \|x\| + 3$

Shift towards -ve y- axis



$f(x) = \|x\| - 1$	$f(x) = \|x\| - 2$	$f(x) = \|x\| - 3$

PYQ AND EXPECTED QUESTION

1. The function f is defined by (IMP-2012)

$$f(x)=\begin{cases} 2-x, & x<0 \\ 2, & x=0 \\ 2+x, & x>0 \end{cases}$$

Draw the graph of Find f(x)

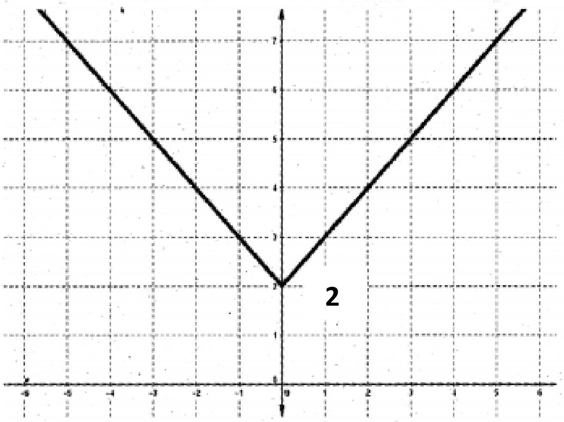
ANS:

For $x < 0$

x	-1	-2
$2-x$	3	4

For $x > 0$

x	1	2
$2+x$	3	4



2.

i) If A = {2,4}, B = {1,3,5}. Then the number of relations from A to B is ………… (IMP-2013)

ii) If P={-1,1}, form the set P x P x P

ANS:

i) number of relations from A to B $2^{2 \times 3} = 2^6 = 64$

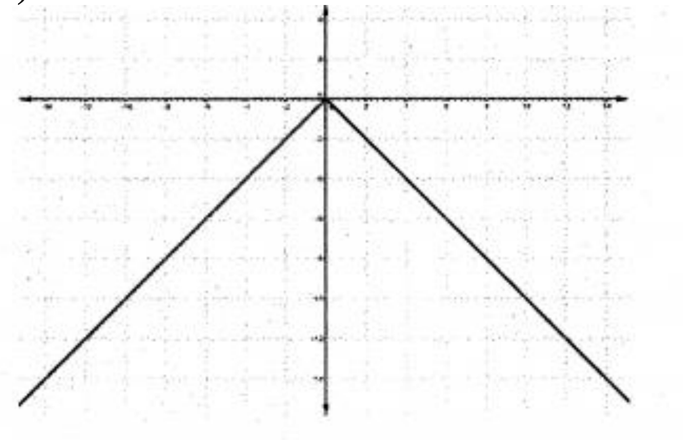
ii) P x P = {-1,1} x {-1,1} – {(-1,-1),(-1,1), (1-1), (1,1)}

P x P x P={(-1,-1), (-1,1), (1,-1),(1,1)} x {-1,1}

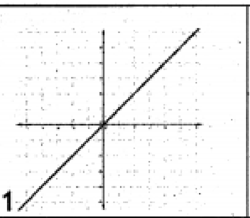
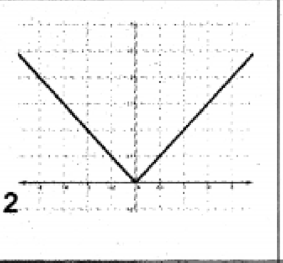
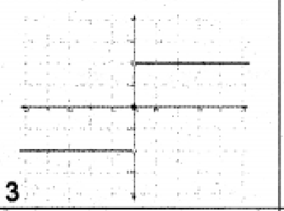
= {(-1,-1,-1),(-1-1,1),(-1,1,-1),(-1,1,1), (1,-1,-1), (1,-1,1), (1,1-1), (1,1,1)}

3. Consider the function f : **R** → **R** defined by f(x) = -|x| (IMP-2013)
   i) Find the domain and range of f.
   ii) Draw the graph of f.

   ANS:
   i) Domain = R; Range = (-∝,0]
   ii)

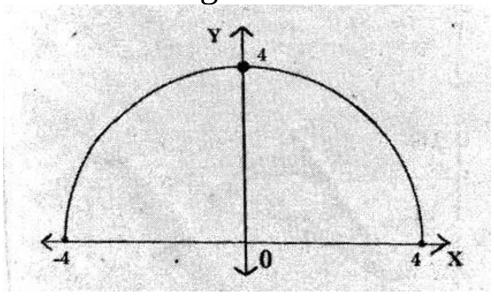


4. Match the following (IMP-2015)

 1	**Modulus function.** $f: R \to R; f(x) = \lvert x \rvert$
 2	**Signum function** $f: R \to R$ $f(x) = \begin{cases} \frac{\lvert x \rvert}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
 3	**Identity function** $f: R \to R; f(x) = x$
	Greatest integer function. $f: R \to R\ f(x) = [x]$

ANS:
1- Identity function, f :R →R : f(x) = x
2- Modulus function, f :R →R : f(x) = |x|
3 – Signum function f :R → R

5. The figure shows the graph of a function $f(x)$ which is a semi-circle centred at origin **MARCH 2018**



a) Write the domain and range of $f(x)$

b) Define the function $f(x)$.

ANS:

Domain = [-4,4] (x-axis)

Range = [0,4] (y-axis)

$x^2+y^2=16$

$y^2 = 16 - x^2$

$y = \sqrt{16 - x^2} = f(x)$