3. TRIGONOMETRIC FUNCTIONS

Trigonometric Functions

Trigonometric ratios are defined for acute angles as the ratio of the sides of a right angled triangle. The extension of trigonometric ratios to any angle in terms of radian measure (real numbers) are called trigonometric functions. The signs of trigonometric functions in different quadrants have been given in the following table:

	I	II	III	IV
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
$\text{cosec } x$	+	+	–	–
$\sec x$	+	–	–	+
$\cot x$	+	–	+	–

Domain and range of trigonometric functions

Functions	Domain	Range
sine	**R**	[–1, 1]
cosine	**R**	[–1, 1]
tan	$\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$	**R**
cot	$\mathbf{R} - \{n\pi : n \in \mathbf{Z}\}$	**R**
sec	$\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$	$\mathbf{R} - (-1, 1)$
cosec	$\mathbf{R} - \{n\pi : n \in \mathbf{Z}\}$	$\mathbf{R} - (-1, 1)$

Sine, cosine and tangent of some angles less than 90°

	0°	15°	18°	30°	36°	45°	60°	90°
sine	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\frac{1}{\sqrt{3}}$	$\sqrt{5-2\sqrt{5}}$	1	$\sqrt{3}$	not defined

Allied or related angles

The angles $\frac{n\pi}{2} \pm \theta$ are called allied or related angles and $\theta \pm n \times 360°$ are called coterminal angles. For general reduction, we have the following rules. The value of any trigonometric function for $\frac{n\pi}{2} \pm \theta$ is numerically equal to

(a) the value of the same function if n is an even integer with algebraic sign of the function as per the quadrant in which angles lie.

(b) corresponding cofunction of θ if n is an odd integer with algebraic sign of the function for the quadrant in which it lies. Here sine and cosine; tan and cot; sec and cosec are cofunctions of each other.

3.1.7 *Functions of negative angles*

Let θ be any angle. Then

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$, $\cot(-\theta) = -\cot\theta$

$\sec(-\theta) = \sec\theta$, $\text{cosec}(-\theta) = -\text{cosec}\,\theta$

Some formulae regarding compound angles

An angle made up of the sum or differences of two or more angles is called a compound angle. The basic results in this direction are called trigonometric identities as given below:

(i) sin (A + B) = sin A cos B + cos A sin B

(ii) sin (A – B) = sin A cos B – cos A sin B

(iii) cos (A + B) = cos A cos B – sin A sin B

(iv) cos (A – B) = cos A cos B + sin A sin B

(v) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$

(viii) $\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$

(ix) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

(x) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \frac{1-\tan^2 A}{1+\tan^2 A}$

(xi) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

(xii) $\sin 3A = 3\sin A - 4\sin^3 A$

(xiii) $\cos 3A = 4\cos^3 A - 3\cos A$

(xiv) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

(xv) $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

(xvi) $\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$

(xvii) $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

(xviii) $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

(xix) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(xx) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(xxi) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(xxii) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(xxiii) $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$ $\begin{bmatrix} + \text{ if } \frac{A}{2} \text{ lies in quadrants I or II} \\ - \text{ if } \frac{A}{2} \text{ lies in III or IV quadrants} \end{bmatrix}$

(xxiv) $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$ $\begin{bmatrix} + \text{ if } \frac{A}{2} \text{ lies in I or IV quadrants} \\ - \text{ if } \frac{A}{2} \text{ lies in II or III quadrants} \end{bmatrix}$

(xxv) $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$ $\begin{bmatrix} + \text{ if } \frac{A}{2} \text{ lies in I or III quadrants} \\ - \text{ if } \frac{A}{2} \text{ lies in II or IV quadrants} \end{bmatrix}$

ALL TRIGONOMETRIC FUNCTIONS GRAPHS

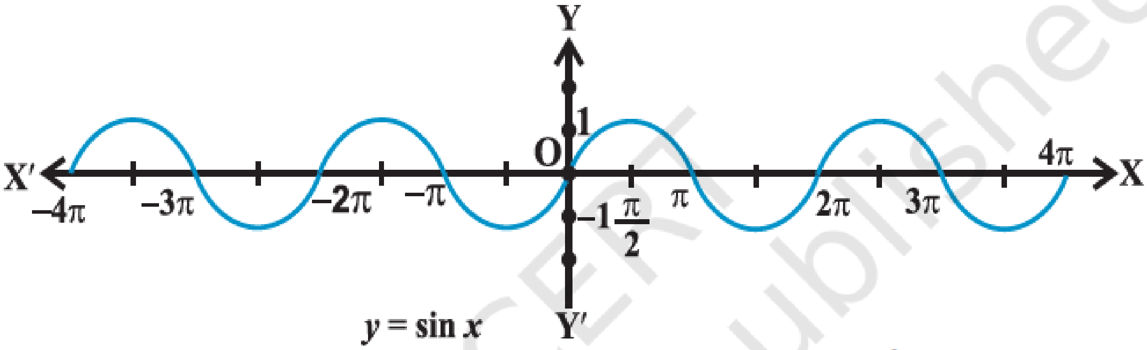


Fig 3.8

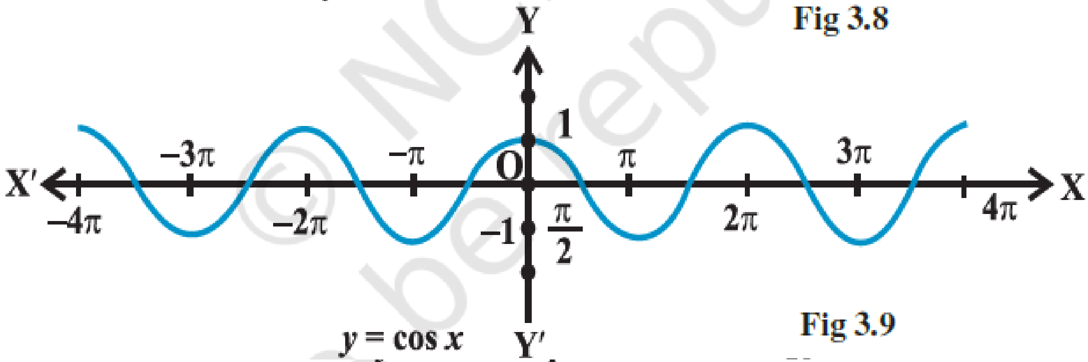


Fig 3.9

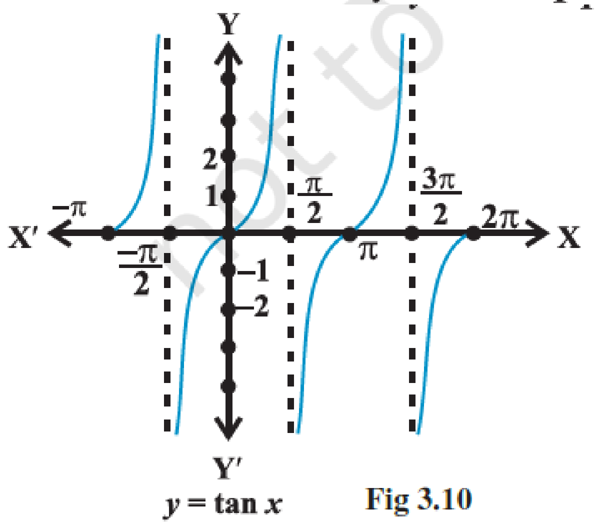


Fig 3.10

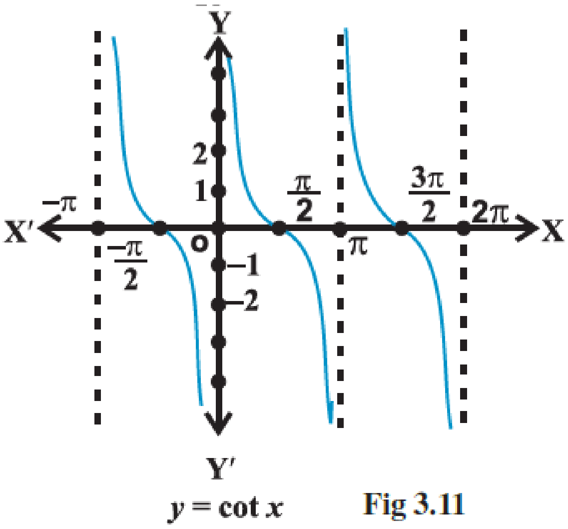
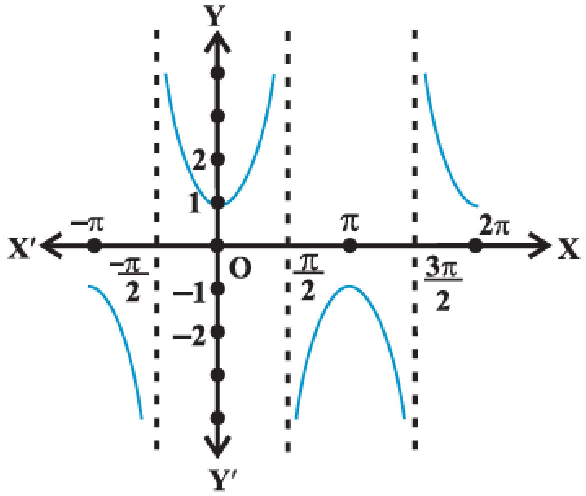
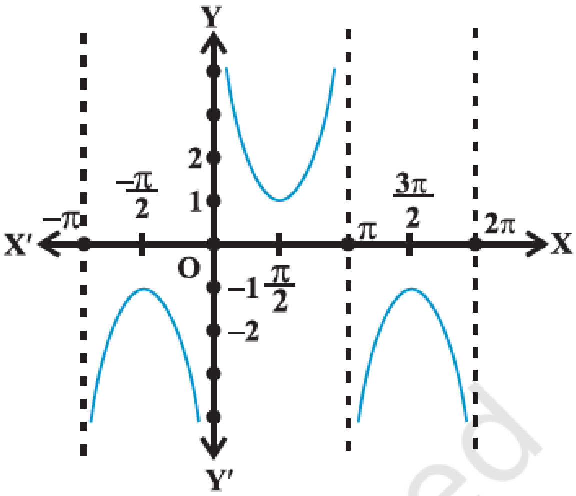


Fig 3.11



y = sec x

Fig 3.12



y = cosec x

Fig 3.13

Example 8 Find the value of $\sin \frac{31\pi}{3}$.

Solution

$$\sin \frac{31\pi}{3} = \sin \left(10\pi + \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example 9 Find the value of cos (–1710°).

Solution We know that values of cos x repeats after an interval of 2π or 360°. Therefore, $\cos(-1710°) = \cos(-1710° + 5 \times 360°)$

$$= \cos(-1710° + 1800°) = \cos 90° = 0.$$

Example 10. Find the values of the trigonometric function sin 765°

Solution

$$\therefore \sin 765° = \sin(2 \times 360° + 45°) = \sin 45° = \frac{1}{\sqrt{2}}$$

Example 11. Find the values of the trigonometric function cosec (– 1410°)

Solution

$$\therefore \operatorname{cosec}(-1410°) = \operatorname{cosec}(-1410° + 4 \times 360°)$$

$$= \operatorname{cosec}(-1410° + 1440°)$$

$$= \operatorname{cosec} 30° = 2$$

Example 11. Find the values of the trigonometric function $\tan\frac{19\pi}{3}$

Solution

$$\therefore \tan\frac{19\pi}{3} = \tan 6\frac{1}{3}\pi = \tan\left(6\pi + \frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \tan 60° = \sqrt{3}$$

Example 10 Prove that

$$3\sin\frac{\pi}{6}\sec\frac{\pi}{3}-4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}=1$$

Solution We have

$$\text{L.H.S.} = 3\sin\frac{\pi}{6}\sec\frac{\pi}{3}-4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}$$

$$= 3\times\frac{1}{2}\times 2 - 4\sin\left(\pi-\frac{\pi}{6}\right)\times 1 = 3-4\sin\frac{\pi}{6}$$

$$= 3-4\times\frac{1}{2} = 1 = \text{R.H.S.}$$

Example 11 Find the value of sin 15°.

Solution We have

$$\sin 15^\circ = \sin(45^\circ-30^\circ)$$

$$= \sin 45^\circ\cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Example 12 Find the value of $\tan\frac{13\pi}{12}$.

Solution We have

$$\tan\frac{13\pi}{12} = \tan\left(\pi+\frac{\pi}{12}\right) = \tan\frac{\pi}{12}=\tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$= \frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}$$

Example 13 Prove that

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

Solution We have

$$\text{L.H.S.} = \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$$

Dividing the numerator and denominator by cos x cos y, we get

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

Example 14 Show that

$$\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$$

Solution We know that $3x = 2x + x$

Therefore, $\tan 3x = \tan(2x + x)$

or $$\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

or $\tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x$

or $\tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$

or $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$.

Example 15 Prove that

$$\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2}\cos x$$

Solution Using the Identity 20(i), we have

L.H.S. $= \cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$

$$= 2\cos\left(\frac{\frac{\pi}{4}+x+\frac{\pi}{4}-x}{2}\right)\cos\left(\frac{\frac{\pi}{4}+x-(\frac{\pi}{4}-x)}{2}\right)$$

$$= 2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{R.H.S.}$$

Example 16 Prove that $\frac{\cos 7x+\cos 5x}{\sin 7x-\sin 5x}=\cot x$

Solution Using the Identities 20 (i) and 20 (iv), we get

L.H.S. $= \frac{2\cos\frac{7x+5x}{2}\cos\frac{7x-5x}{2}}{2\cos\frac{7x+5x}{2}\sin\frac{7x-5x}{2}} = \frac{\cos x}{\sin x} = \cot x = \text{R.H.S.}$

Example 17 Prove that $=\frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x}=\tan x$

Solution We have

L.H.S. $= \frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \frac{\sin 5x+\sin x-2\sin 3x}{\cos 5x-\cos x}$

$$= \frac{2\sin 3x\cos 2x-2\sin 3x}{-2\sin 3x\sin 2x} = -\frac{\sin 3x(\cos 2x-1)}{\sin 3x\sin 2x}$$

$$= \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \tan x = \text{R.H.S.}$$

PYQ & EXPECTED QUESTIONS

Q1) Prove the following:

16. $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\dfrac{\sin 2x}{\cos 10x}$

17. $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$

18. $\dfrac{\sin x - \sin y}{\cos x + \cos y} = \tan \dfrac{x-y}{2}$

19. $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

20. $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$

21. $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

Q2) Find the value of: (i) sin 75° (ii) tan 15°

Q3) Which of the following is equal to 520^0?

i. $\frac{26\pi}{9}$ ii. 9π iii. 26π iv. $\frac{9\pi}{26}$

Q4) The value of $\sin(\pi - x) = \ldots\ldots\ldots$

Q5) Show that $\tan 15^0 = 2-\sqrt{3}$