4. PRINCIPLE OF MATHEMATICAL INDUCTION(PMI)

The principle of mathematical induction

Let P(*n*) be a given statement involving the natural number *n* such that
(i) The statement is true for $n = 1$, i.e., P(1) is true (or true for any fixed natural number) and
(ii) If the statement is true for $n = k$ (where *k* is a particular but arbitrary natural number), then the statement is also true for $n = k + 1$, i.e, truth of P(*k*) implies the truth of P(*k* + 1). Then P(*n*) is true for all natural numbers *n*.

Example 1 For all $n \geq 1$, prove that

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution Let the given statement be P(*n*), i.e.,

$$P(n):\ 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n = 1$, $P(1):\ 1 = \frac{1(1+1)(2\times1+1)}{6} = \frac{1\times2\times3}{6} = 1$ which is true.

Assume that P(*k*) is true for some positive integer *k*, i.e.,

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \ldots (1)$$

We shall now prove that P(*k* + 1) is also true. Now, we have

$$(1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2) + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{[Using (1)]}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+1+1)\{2(k+1)+1\}}{6}$$

Thus P(*k* + 1) is true, whenever P(*k*) is true.

Hence, from the principle of mathematical induction, the statement P(*n*) is true for all natural numbers *n*.

Example 3 For all $n \geq 1$, prove that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

Solution We can write

$$P(n): \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

We note that $P(1): \frac{1}{1.2}=\frac{1}{2}=\frac{1}{1+1}$, which is true. Thus, P(n) is true for $n = 1$.

Assume that P(k) is true for some natural number k,

i.e., $$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad ... (1)$$

We need to prove that P(k + 1) is true whenever P(k) is true. We have

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad \text{[Using (1)]}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{(k^2+2k+1)}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}=\frac{k+1}{(k+1)+1}$$

Thus P(k + 1) is true whenever P(k) is true. Hence, by the principle of mathematical induction, P(n) is true for all natural numbers.

Most repeated question (Example:4)

Example 4 For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.

Solution We can write

P(n) : $7^n - 3^n$ is divisible by 4.

We note that

P(1): $7^1 - 3^1 = 4$ which is divisible by 4. Thus P(n) is true for $n = 1$

Let P(k) be true for some natural number k,

i.e., P(k) : $7^k - 3^k$ is divisible by 4.

We can write $7^k - 3^k = 4d$, where $d \in$ N.

Now, we wish to prove that P(k + 1) is true whenever P(k) is true.

Now $7^{(k+1)} - 3^{(k+1)} = 7^{(k+1)} - 7.3^k + 7.3^k - 3^{(k+1)}$

$= 7(7^k - 3^k) + (7 - 3)3^k = 7(4d) + (7 - 3)3^k$

$= 7(4d) + 4.3^k = 4(7d + 3^k)$

From the last line, we see that $7^{(k+1)} - 3^{(k+1)}$ is divisible by 4. Thus, P(k + 1) is true when P(k) is true. Therefore, by principle of mathematical induction the statement is true for every positive integer n.

Q) For any natural number n, $7^n - 2^n$ is divisible by 5.

Let P(n): $7^n - 2^n$ is divisible by 5, for any natural number n.
Now, P(l) = $7^1 - 2^1 = 5$, which is divisible by 5.
Hence, P(l) is true.
Let us assume that, P(n) is true for some natural number n = k.
∴ P(k) = $7^k - 2^k$ is divisible by 5
or $7^k - 2^k = 5m$, m∈ N (i)
Now, we have to prove that P(k + 1) is true.
P(k+ 1): $7^{k+1} - 2^{k+1}$
$= 7^k \cdot 7 - 2^k \cdot 2$
$= (5 + 2)7^k - 2^k \cdot 2$
$= 5.7^k + 2.7^k - 2 \cdot 2^k$
$= 5.7^k + 2(7^k - 2^k)$
$= 5 \cdot 7^k + 2(5m)$ (using (i))
$= 5(7^k + 2m)$, which divisible by 5.
Thus, P(k + 1) is true whenever P(k) is true.
So, by the principle of mathematical induction P(n) is true for all natural numbers

Example 6 Prove that

$2.7^n + 3.5^n - 5$ is divisible by 24, for all $n \in \mathbf{N}$.

Solution Let the statement P(n) be defined as

P(n) : $2.7^n + 3.5^n - 5$ is divisible by 24.

We note that P(n) is true for $n = 1$, since $2.7 + 3.5 - 5 = 24$, which is divisible by 24.

Assume that P(k) is true

i.e. $2.7^k + 3.5^k - 5 = 24q$, when $q \in \mathbf{N}$ ... (1)

Now, we wish to prove that P($k + 1$) is true whenever P(k) is true.

We have

$$\begin{aligned}
2.7^{k+1} + 3.5^{k+1} - 5 &= 2.7^k . 7^1 + 3.5^k . 5^1 - 5 \\
&= 7\,[2.7^k + 3.5^k - 5 - 3.5^k + 5] + 3.5^k . 5 - 5 \\
&= 7\,[24q - 3.5^k + 5] + 15.5^k - 5 \\
&= 7 \times 24q - 21.5^k + 35 + 15.5^k - 5 \\
&= 7 \times 24q - 6.5^k + 30 \\
&= 7 \times 24q - 6\,(5^k - 5) \\
&= 7 \times 24q - 6\,(4p) \quad [(5^k - 5) \text{ is a multiple of 4 (why?)}] \\
&= 7 \times 24q - 24p \\
&= 24\,(7q - p) \\
&= 24 \times r;\ r = 7q - p, \text{ is some natural number.} \quad \text{... (2)}
\end{aligned}$$

The expression on the R.H.S. of (1) is divisible by 24. Thus P($k + 1$) is true whenever P(k) is true.

Hence, by principle of mathematical induction, P(n) is true for all $n \in \mathbf{N}$.

PYQ & EXPECTED QUESTIONS

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$:

1. $1 + 3 + 3^2 + ... + 3^{n-1} = \dfrac{(3^n - 1)}{2}$.

2. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$.

3. $1 + \dfrac{1}{(1+2)} + \dfrac{1}{(1+2+3)} + ... + \dfrac{1}{(1+2+3+...n)} = \dfrac{2n}{(n+1)}$.

4. $1.2.3 + 2.3.4 + \ldots + n(n+1)\,(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

5. $41^n - 14^n$ is a multiple of 27