5. COMPLEX NUMBERS AND QUADRATIC EQUATIONS

Complex numbers

(a) A number which can be written in the form $a + ib$, where a, b are real numbers and $i = -1$ is called a complex number.

(b) If $z = a + ib$ is the complex number, then a and b are called real and imaginary parts, respectively, of the complex number and written as $\text{R}e\,(z) = a$, $\text{I}m\,(z) = b$.

(c) Order relations "greater than" and "less than" are not defined for complex numbers.

(d) If the imaginary part of a complex number is zero, then the complex number is known as purely real number and if real part is zero, then it is called purely imaginary number, for example, 2 is a purely real number because its imaginary part is zero and $3i$ is a purely imaginary number because its real part is zero.

Algebra of complex numbers

(a) Two complex numbers $z1 = a + ib$ and $z2 = c + id$ are said to be equal if $a = c$ and $b = d$.

(b) Let $z1 = a + ib$ and $z2 = c + id$ be two complex numbers then
$z1 + z2 = (a + c) + i\,(b + d)$.

Addition of complex numbers satisfies the following properties

1. As the sum of two complex numbers is again a complex number, the set of complex numbers is closed with respect to addition.
2. Addition of complex numbers is commutative, i.e., $z1 + z2 = z2 + z1$
3. Addition of complex numbers is associative, i.e., $(z1 + z2) + z3 = z1 + (z2 + z3)$
4. For any complex number $z = x + i\,y$, there exist 0, i.e., $(0 + 0i)$ complex number such that $z + 0 = 0 + z = z$, known as identity element for addition.
5. For any complex number $z = x + iy$, there always exists a number $-z = -a - ib$ such that $z + (-z) = (-z) + z = 0$ and is known as the additive inverse of z.

Multiplication of complex numbers

Let $z1 = a + ib$ and $z2 = c + id$, be two complex numbers. Then
$z1 \,.\, z2 = (a + ib)\,(c + id) = (ac - bd) + i\,(ad + bc)$

1. As the product of two complex numbers is a complex number, the set of complex numbers is closed with respect to multiplication.
2. Multiplication of complex numbers is commutative, i.e., $z1.z2 = z2.z1$
3. Multiplication of complex numbers is associative, i.e., $(z1.z2)\,.\,z3 = z1\,.\,(z2.z3)$

4. For any complex number $z = x + iy$, there exists a complex number 1, i.e., $(1 + 0i)$ such that
   $z \cdot 1 = 1 \cdot z = z$, known as identity element for multiplication.
5. For any non zero complex number $z = x + i\,y$, there exists a complex number $\frac{1}{z}$ such that $z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1$, i.e., multiplicative inverse of $a + ib = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$.
6. For any three complex numbers z_1, z_2 and z_3,
   $$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$
   and $$(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$$
   i.e., for complex numbers multiplication is distributive over addition.

5.1.7 Let $z_1 = a + ib$ and $z_2 (\neq 0) = c + id$. Then

$$z_1 \div z_2 = \frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

Conjugate of a complex number

Let $z = a + ib$ be a complex number. Then a complex number obtained by changing the sign of imaginary part of the complex number is called the conjugate of z and it is denoted by $\bar{z}$, i.e., $\bar{z} = a - ib$.

Note that additive inverse of z is $-a - ib$ but conjugate of z is $a - ib$.

We have:

1. $\overline{(\bar{z})} = z$
2. $z + \bar{z} = 2\,\text{Re}\,(z)$, $z - \bar{z} = 2\,i\,\text{Im}(z)$
3. $z = \bar{z}$, if z is purely real.
4. $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
5. $z \cdot \bar{z} = \{\text{Re}\,(z)\}^2 + \{\text{Im}\,(z)\}^2$.
6. $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$, $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$
7. $\overline{(z_1 \cdot z_2)} = (\bar{z}_1)\,(\bar{z}_2)$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\bar{z}_1)}{(\bar{z}_2)}$ $(\bar{z}_2 \neq 0)$

5.3.5 *Power of i* we know that

$$i^3 = i^2 i = (-1)\, i = -i\,, \qquad i^4 = \left(i^2\right)^2 = (-1)^2 = 1$$

$$i^5 = \left(i^2\right)^2 i = (-1)^2\, i = i\,, \qquad i^6 = \left(i^2\right)^3 = (-1)^3 = -1\,, \text{ etc.}$$

Also, we have $i^{-1} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i, \quad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{1} = i, \quad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

In general, for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

5.3.7 *Identities* We prove the following identity

$\left(z_1 + z_2\right)^2 = z_1^2 + z_2^2 + 2z_1 z_2$, for all complex numbers z_1 and z_2.

Proof We have, $(z_1 + z_2)^2 = (z_1 + z_2)\,(z_1 + z_2)$,

$= (z_1 + z_2)\, z_1 + (z_1 + z_2)\, z_2$ (Distributive law)

$= z_1^2 + z_2 z_1 + z_1 z_2 + z_2^2$ (Distributive law)

$= z_1^2 + z_1 z_2 + z_1 z_2 + z_2^2$ (Commutative law of multiplication)

$= z_1^2 + 2z_1 z_2 + z_2^2$

Similarly, we can prove the following identities:

(i) $\left(z_1 - z_2\right)^2 = z_1^2 - 2\, z_1\, z_2 + z_2^2$

(ii) $\left(z_1 + z_2\right)^3 = z_1^3 + 3\, z_1^2 z_2 + 3z_1 z_2^2 + z_2^3$

(iii) $\left(z_1 - z_2\right)^3 = z_1^3 - 3z_1^2 z_2 + 3z_1 z_2^2 - z_2^3$

(iv) $z_1^2 - z_2^2 = \left(z_1 + z_2\right)\left(z_1 - z_2\right)$

In fact, many other identities which are true for all real numbers, can be proved to be true for all complex numbers.

Example 2 Express the following in the form of $a + bi$:

(i) $(-5i)\left(\frac{1}{8}i\right)$ (ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3$

Solution (i) $(-5i)\left(\frac{1}{8}i\right) = \frac{-5}{8}i^2 = \frac{-5}{8}(-1) = \frac{5}{8} = \frac{5}{8} + i0$

(ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3 = 2\times\frac{1}{8\times8\times8}\times i^5 = \frac{1}{256}\left(i^2\right)^2 i = \frac{1}{256}i$.

Example 3 Express $(5 - 3i)^3$ in the form $a + ib$.

Solution We have, $(5 - 3i)^3 = 5^3 - 3\times5^2\times(3i) + 3\times5\,(3i)^2 - (3i)^3$

$= 125 - 225i - 135 + 27i = -10 - 198i.$

Example 4 Express $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right)$ in the form of $a + ib$

Solution We have, $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right) = \left(-\sqrt{3}+\sqrt{2}\,i\right)\left(2\sqrt{3}-i\right)$

$= -6+\sqrt{3}i+2\sqrt{6}i-\sqrt{2}\,i^2 = \left(-6+\sqrt{2}\right)+\sqrt{3}\left(1+2\sqrt{2}\right)i$

5.4 The Modulus and the Conjugate of a Complex Number

Let $z = a + ib$ be a complex number. Then, the modulus of z, denoted by $|z|$, is defined to be the non-negative real number $\sqrt{a^2+b^2}$, i.e., $|z| = \sqrt{a^2+b^2}$ and the conjugate of z, denoted as $\bar{z}$, is the complex number $a - ib$, i.e., $\bar{z} = a - ib$.

For example, $|3+i| = \sqrt{3^2+1^2} = \sqrt{10}$, $|2-5i| = \sqrt{2^2+(-5)^2} = \sqrt{29}$,

and $\overline{3+i} = 3-i$, $\overline{2-5i} = 2+5i$, $\overline{-3i-5} = 3i-5$

Observe that the multiplicative inverse of the non-zero complex number z is given by

$$z^{-1} = \frac{1}{a+ib} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$$

or $$z\bar{z} = |z|^2$$

Furthermore, the following results can easily be derived.

For any two compex numbers z_1 and z_2, we have

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $|z_2| \neq 0$

(iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (iv) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ (v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ provided $z_2 \neq 0$.

Example 5 Find the multiplicative inverse of $2 - 3i$.

Solution Let $z = 2 - 3i$

Then $\bar{z} = 2 + 3i$ and $|z|^2 = 2^2 + (-3)^2 = 13$

Therefore, the multiplicative inverse of $2 - 3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

The above working can be reproduced in the following manner also,

$$z^{-1} = \frac{1}{2-3i} = \frac{2+3i}{(2-3i)(2+3i)}$$

$$= \frac{2+3i}{2^2-(3i)^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

Example 6 Express the following in the form $a + ib$

(i) $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$ (ii) i^{-35}

Solution (i) We have, $\frac{5+\sqrt{2}i}{1-\sqrt{2}i} = \frac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \frac{1+\sqrt{2}i}{1+\sqrt{2}i} = \frac{5+5\sqrt{2}i+\sqrt{2}i-2}{1-\left(\sqrt{2}i\right)^2}$

$$= \frac{3+6\sqrt{2}i}{1+2} = \frac{3(1+2\sqrt{2}i)}{3} = 1+2\sqrt{2}i.$$

(ii) $i^{-35} = \frac{1}{i^{35}} = \frac{1}{\left(i^2\right)^{17} i} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = i$

PYQ & EXPECTED QUESTIONS

Q) Express each of the complex number given in the Exercises 1 to 6 in the form $a + ib$.

1. $(5i)\left(-\frac{3}{5}i\right)$ 2. $i^9 + i^{19}$ 3. i^{-39} 4. $3(7 + i7) + i(7 + i7)$

5. $(1 - i) - (-1 + i6)$ 6. $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$

Q) Find the multiplicative inverse of each of the complex numbers given

a. $4 - 3i$
b. $5 + 3i$
c. $-i$

Q) Express the following expression in the form of $a + ib$:

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}\,i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

Q) If a complex number lies in the third quadrant, then its conjugate lies in the _______.

Q) The multiplicative inverse of the complex number 3+4i = _______

Q) i^{18} = ______________

i) 1 ii) 0 iii) -1 iv) i

Q) Express $\frac{1+i}{1-i}$ in the form a+ib.

Q) Express $\frac{2+i}{2-i}$ in the form a+ib.

Q) Express the complex number $\frac{3-\sqrt{-16}}{1-\sqrt{-9}}$ in the form.