8. BINOMIAL THEOREM

Binomial Theorem for Positive Integral Indices

Let us have a look at the following identities done earlier:

$(a+b)^0 = 1 \qquad a+b \neq 0$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = (a+b)^3(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

In these expansions, we observe that

(i) The total number of terms in the expansion is one more than the index. For example, in the expansion of $(a+b)^2$, number of terms is 3 whereas the index of $(a+b)^2$ is 2.

(ii) Powers of the first quantity 'a' go on decreasing by 1 whereas the powers of the second quantity 'b' increase by 1, in the successive terms.

(iii) In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of $a+b$.

Binomial theorem for any positive integer n,

$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ... + {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$

Observations

1. The notation $\sum_{k=0}^{n} {}^nC_k\, a^{n-k}b^k$ stands for

${}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + ...+ {}^nC_ra^{n-r}b^r + ...+{}^nC_na^{n-n}b^n$, where $b^0 = 1 = a^{n-n}$.

Hence the theorem can also be stated as

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k\, a^{n-k}b^k .$$

2. The coefficients ${}^nC_r$ occuring in the binomial theorem are known as binomial coefficients.

3. There are $(n+1)$ terms in the expansion of $(a+b)^n$, i.e., one more than the index.

4. In the successive terms of the expansion the index of a goes on decreasing by unity. It is n in the first term, $(n-1)$ in the second term, and so on ending with zero in the last term. At the same time the index of b increases by unity, starting with zero in the first term, 1 in the second and so on ending with n in the last term.

5. In the expansion of $(a+b)^n$, the sum of the indices of a and b is $n + 0 = n$ in the first term, $(n-1) + 1 = n$ in the second term and so on $0 + n = n$ in the last term. Thus, it can be seen that the sum of the indices of a and b is n in every term of the expansion.

8.1.5 The p^{th} term from the end

The p^{th} term from the end in the expansion of $(a + b)^n$ is $(n - p + 2)^{th}$ term from the beginning.

8.1.6 Middle terms

The middle term depends upon the value of n.

(a) If n is even: then the total number of terms in the expansion of $(a + b)^n$ is $n + 1$ (odd). Hence, there is only one middle term, i.e., $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term.

(b) If n is odd: then the total number of terms in the expansion of $(a + b)^n$ is $n + 1$ (even). So there are two middle terms i.e., $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ are two middle terms.

8.1.7 Binomial coefficient

In the Binomial expression, we have

$$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_n\, b^n \qquad \ldots (1)$$

The coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ... , ${}^nC_n$ are known as binomial or combinatorial coefficients.

Putting $a = b = 1$ in (1), we get

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$$

Thus the sum of all the binomial coefficients is equal to 2^n.

Again, putting $a = 1$ and $b = -1$ in (i), we get

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots$$

Thus, the sum of all the odd binomial coefficients is equal to the sum of all the even binomial coefficients and each is equal to $\frac{2^n}{2} = 2^{n-1}$.

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$$

Example 1 Expand $\left(x^2+\frac{3}{x}\right)^4, x \neq 0$

Solution By using binomial theorem, we have

$$\left(x^2+\frac{3}{x}\right)^4 = {}^4C_0(x^2)^4 + {}^4C_1(x^2)^3\left(\frac{3}{x}\right) + {}^4C_2(x^2)^2\left(\frac{3}{x}\right)^2 + {}^4C_3(x^2)\left(\frac{3}{x}\right)^3 + {}^4C_4\left(\frac{3}{x}\right)^4$$

$$= x^8 + 4.x^6 . \frac{3}{x} + 6.x^4 . \frac{9}{x^2} + 4.x^2 . \frac{27}{x^3} + \frac{81}{x^4}$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}.$$

Example 2 Compute $(98)^5$.

Solution We express 98 as the sum or difference of two numbers whose powers are easier to calculate, and then use Binomial Theorem.

Write $98 = 100 - 2$

Therefore, $(98)^5 = (100-2)^5$

$$= {}^5C_0\,(100)^5 - {}^5C_1\,(100)^4.2 + {}^5C_2\,(100)^3 2^2$$
$$- {}^5C_3\,(100)^2\,(2)^3 + {}^5C_4\,(100)\,(2)^4 - {}^5C_5\,(2)^5$$
$$= 10000000000 - 5 \times 100000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000$$
$$\times 8 + 5 \times 100 \times 16 - 32$$

$$= 10040008000 - 1000800032 = 9039207968.$$

Example 4 Using binomial theorem, prove that $6^n – 5n$ always leaves remainder 1 when divided by 25.

Solution For two numbers a and b if we can find numbers q and r such that $a = bq + r$, then we say that b divides a with q as quotient and r as remainder. Thus, in order to show that $6^n – 5n$ leaves remainder 1 when divided by 25, we prove that $6^n – 5n = 25k + 1$, where k is some natural number.

We have

$$(1 + a)^n = {}^nC_0 + {}^nC_1 a + {}^nC_2 a^2 + ... + {}^nC_n a^n$$

For $a = 5$, we get

$$(1 + 5)^n = {}^nC_0 + {}^nC_1 5 + {}^nC_2 5^2 + ... + {}^nC_n 5^n$$

i.e. $(6)^n = 1 + 5n + 5^2.{}^nC_2 + 5^3.{}^nC_3 + ... + 5^n$

i.e. $6^n – 5n = 1+5^2 ({}^nC_2 + {}^nC_3 5 + ... + 5^{n-2})$

or $6^n – 5n = 1+ 25 ({}^nC_2 + 5 .{}^nC_3 + ... + 5^{n-2})$

or $6^n – 5n = 25k+1$ where $k = {}^nC_2 + 5 .{}^nC_3 + ... + 5^{n-2}$.

This shows that when divided by 25, $6^n – 5n$ leaves remainder 1.

Example 5

Find $(a + b)^4 – (a – b)^4$. Hence, evaluate $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$

Solution

$$(a+b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4$$

$$(a-b)^4 = {}^4C_0 a^4 - {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 - {}^4C_3 ab^3 + {}^4C_4 b^4$$

$$\therefore (a+b)^4 - (a-b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4$$
$$-\left[{}^4C_0 a^4 - {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 - {}^4C_3 ab^3 + {}^4C_4 b^4\right]$$
$$= 2\left({}^4C_1 a^3 b + {}^4C_3 ab^3\right) = 2\left(4a^3 b + 4ab^3\right)$$
$$= 8ab\left(a^2 + b^2\right)$$

By putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4 = 8\left(\sqrt{3}\right)\left(\sqrt{2}\right)\left\{\left(\sqrt{3}\right)^2 + \left(\sqrt{2}\right)^2\right\}$$
$$= 8\left(\sqrt{6}\right)\{3+2\} = 40\sqrt{6}$$

Example 6

Find $(x+1)^6 + (x-1)^6$. Hence or otherwise evaluate $(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6$

Solution

$$(x+1)^6 = {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_4x^2 + {}^6C_5x + {}^6C_6$$

$$(x-1)^6 = {}^6C_0x^6 - {}^6C_1x^5 + {}^6C_2x^4 - {}^6C_3x^3 + {}^6C_4x^2 - {}^6C_5x + {}^6C_6$$

$$\therefore (x+1)^6 + (x-1)^6 = 2\left[{}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6\right]$$

$$= 2\left[x^6 + 15x^4 + 15x^2 + 1\right]$$

$$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6 = 2\left[\left(\sqrt{2}\right)^6 + 15\left(\sqrt{2}\right)^4 + 15\left(\sqrt{2}\right)^2 + 1\right]$$

$$= 2(8 + 15\times4 + 15\times2 + 1)$$

$$= 2(8 + 60 + 30 + 1)$$

$$= 2(99) = 198$$

General and Middle Terms

1. In the binomial expansion for $(a+b)^n$, we observe that the first term is ${}^nC_0a^n$, the second term is ${}^nC_1a^{n-1}b$, the third term is ${}^nC_2a^{n-2}b^2$, and so on. Looking at the pattern of the successive terms we can say that the $(r+1)^{th}$ term is ${}^nC_ra^{n-r}b^r$. The $(r+1)^{th}$ term is also called the *general term* of the expansion $(a+b)^n$. It is denoted by T_{r+1}. Thus $T_{r+1} = {}^nC_r\, a^{n-r}b^r$

general term $\mathbf{T_{r+1} = {}^nC_r\, a^{n-r}b^r}$.

3. In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, where $x \neq 0$, the middle term is $\left(\frac{2n+1+1}{2}\right)^{th}$ i.e., $(n+1)^{th}$ term, as $2n$ is even.

It is given by ${}^{2n}C_n x^n \left(\frac{1}{x}\right)^n = {}^{2n}C_n$ (constant).

This term is called the *term independent* of x or the constant term.

Example 5 Find a if the 17^{th} and 18^{th} terms of the expansion $(2 + a)^{50}$ are equal.

Solution The $(r + 1)^{th}$ term of the expansion $(x + y)^n$ is given by $T_{r+1} = {}^nC_r x^{n-r} y^r$.

For the 17^{th} term, we have, $r + 1 = 17$, i.e., $r = 16$

Therefore, $T_{17} = T_{16+1} = {}^{50}C_{16}\,(2)^{50-16}\,a^{16}$

$= {}^{50}C_{16}\,2^{34}\,a^{16}$

Similarly, $T_{18} = {}^{50}C_{17}\,2^{33}\,a^{17}$

Given that $T_{17} = T_{18}$

So ${}^{50}C_{16}\,(2)^{34}\,a^{16} = {}^{50}C_{17}\,(2)^{33}\,a^{17}$

Therefore $$\frac{{}^{50}C_{16}\,.\,2^{34}}{{}^{50}C_{17}\,.\,2^{33}} = \frac{a^{17}}{a^{16}}$$

i.e., $$a = \frac{{}^{50}C_{16} \times 2}{{}^{50}C_{17}} = \frac{50!}{16!\,34!} \times \frac{17!\ .\ 33!}{50!} \times 2 = 1$$

PYQ & EXPECTED QUESTIONS

Q)

i) The number of terms in the expansion of $(\frac{x}{3} + 9y)^{10}$ is ______.(IMP-2013)

ii) Find the middle term in the above expansion.

Ans:

i) 10+1 = 11 terms

ii)

Middle term = $t_{\frac{10}{2}+1} = t_6 = 10C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5$

$= 10C_5 x^5 \times 3^{-5} \times 9^5 y^5$

$= 10C_5 x^5 \times 3^{-5} \times 3^{10} y^5 = 10C_5 \times 3^5 x^5 y^5$

Q)

i) Find the general term in the expansion of $(x + y)^n$

ii) Find the middle term in the expansion of $(2x + \frac{1}{3y})^{18}$ (MARCH-2014)

Ans:

i) General term = $t_{r+1} = nC_r (x)^{n-r}(y)^r$

ii) Middle term = $t_{\frac{n}{2}+1} = t_{\frac{18}{2}+1} = t_{10}$

$= 18C_9 (2x)^{18-9}\left(\frac{1}{3y}\right)^9$

$= 18C_9 (2x)^9\left(\frac{1}{3y}\right)^9$

Q)
i) Write the general term in the expansion of $(a + b)^n$
ii) Find the 9th term in the expansion of $(\frac{x}{2} + \frac{6}{x^2})^{12}$ (IMP-2014)
Ans:

i) General term = $t_{r+1} = {}^{12}C_r (a)^{12-r} (b)^r$

ii) $t_9 = {}^{12}C_8 \left(\frac{x}{2}\right)^{12-8} \left(\frac{6}{x^2}\right)^8$

$= {}^{12}C_8 \left(\frac{x}{2}\right)^4 \left(\frac{6}{x^2}\right)^8$

$= {}^{12}C_4 \frac{x^4}{2^4} \times \frac{2^8 \times 3^8}{x^{16}} = {}^{12}C_4 \frac{2^4 \times 3^8}{x^{12}}$

Q)
Consider the expansion of $(\frac{x}{9} + 9y)^{2n}$ (MARCH-2011)
i) The number of terms in the expansion is ______
(a) 2n
(b) n+1
(c) 2n+1
(d) 2n-1
ii) What is its $(n+1)^{th}$ term?
iii) If n = 5, find its middle term.
Ans:

i) c) $2n+1$

ii) $t_{n+1} = 2nC_n \left(\frac{x}{9}\right)^{2n-n} (9y)^n = 2nC_n \left(\frac{x}{9}\right)^n (9y)^n$

$= 2nC_n (x)^n (y)^n$

iii) If $n = 5$, $\Rightarrow 2n = 10$ therefore the middle term will be $\frac{10}{2} + 1 = 6$ the term.

$t_6 = 10C_5 (x)^5 (y)^5$