9. SEQUENCES AND SERIES

A Geometric progression (G.P.) is a sequence in which each term except the first is obtained by multiplying the previous term by a non-zero constant called the **common ratio**. Let us consider a G.P. with first non-zero term a and common ratio r,

i.e., $a, ar, ar^2, \dots, ar^{n-1}, \dots$

Here, common ratio $r = \frac{ar^{n-1}}{ar^{n-2}}$

The **general term** or ***n*th term** of G.P. is given by $a_n = ar^{n-1}$.

Last term l of a G.P. is same as the nth term and is given by $l = ar^{n-1}$.

and the nth term from the last is given by $a_n = \frac{l}{r^{n-1}}$

The sum Sn of the first n terms is given by

$S_n = \frac{a(r^n-1)}{r-1}$ if $r \neq 1$

$S_n = na$ if $r = 1$

If a, G and b are in G.P., then G is called the **geometric mean** of the numbers a and b and is given by

$G = \sqrt{ab}$

(i) If the terms of a G.P. are multiplied or divided by the same non-zero constant ($k \neq 0$), they still remain in G.P.

If $a_1, a_2, a_3, \dots$, are in G.P., then $a_1 k, a_2 k, a_3 k, \dots$ and $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \dots$

are also in G.P. with same common ratio, in particularly

if $a_1, a_2, a_3, \dots$ are in G.P., then $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots$ are also in G.P.

(ii) If $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ are two G.P.s, then $a_1 b_1, a_2 b_2, a_3 b_3, \dots$ and

$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots$ are also in G.P.

(iii) If $a_1, a_2, a_3, \dots$ are in A.P. ($a_i > 0 \ \forall \ i$),

then $x^{a1}, x^{a2}, x^{a3}, \dots$, are in G.P. ($\forall \ x > 0$)

(iv) If $a_1, a_2, a_3, \dots, a_n$ are in G.P., then $a_1 a_n = a_2 a^{n-1} = a_3 a^{n-2} = \dots$

Example 9 Find the 10th and nth terms of the G.P. 5, 25,125,… .

Solution Here $a = 5$ and $r = 5$. Thus, $a_{10} = 5(5)^{10-1} = 5(5)^9 = 5^{10}$

and $a_n = ar^{n-1} = 5(5)^{n-1} = 5^n$.

Example10 Which term of the G.P., 2,8,32, ... up to n terms is 131072?
Solution Let 131072 be the nth term of the given G.P. Here $a = 2$ and $r = 4$.
Therefore $131072 = a_n = 2(4)^{n-1}$ or $65536 = 4^{n-1}$
This gives $4^8 = 4^{n-1}$.
So that $n - 1 = 8$, i.e., $n = 9$. Hence, 131072 is the 9th term of the G.P.
Example11 In a G.P., the 3rd term is 24 and the 6th term is 192.Find the 10th term.
Solution Here, $a_3 = ar^2 = 24$ ... (1)
And $a_6 = ar^5 = 92$ ... (2)
Dividing (2) by (1), we get $r = 2$. Substituting $r = 2$ in (1), we get $a = 6$.
Hence $a_{10} = 6\,(2)^9 = 3072$.
Example12 Find the sum of first n terms and the sum of first 5 terms of the geometric series $1+\frac{2}{3}+\frac{4}{9}+...$

Solution Here $a = 1$ and $r = \frac{2}{3}$. Therefore

$$S_n = \frac{a\,(1-r^n)}{1-r} = \frac{\left[1-\left(\frac{2}{3}\right)^n\right]}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

In particular, $S_5 = 3\left[1-\left(\frac{2}{3}\right)^5\right] = 3\times\frac{211}{243} = \frac{211}{81}$.

Example 13 How many terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}, \ldots$ are needed to give the sum $\frac{3069}{512}$?

Solution Let n be the number of terms needed. Given that $a = 3$, $r = \frac{1}{2}$ and $S_n = \frac{3069}{512}$

Since $$S_n = \frac{a(1-r^n)}{1-r}$$

Therefore $$\frac{3069}{512} = \frac{3(1-\frac{1}{2^n})}{1-\frac{1}{2}} = 6\left(1-\frac{1}{2^n}\right)$$

or $$\frac{3069}{3072} = 1-\frac{1}{2^n}$$

or $$\frac{1}{2^n} = 1-\frac{3069}{3072} = \frac{3}{3072} = \frac{1}{1024}$$

or $$2^n = 1024 = 2^{10}, \text{ which gives } n = 10.$$

Example 14 The sum of first three terms of a G.P. is $\frac{13}{12}$ and their product is – 1. Find the common ratio and the terms.

Solution Let $\frac{a}{r}$, a, ar be the first three terms of the G.P. Then

$$\frac{a}{r} + ar + a = \frac{13}{12} \quad \ldots (1)$$

and $$\left(\frac{a}{r}\right)(a)(ar) = -1 \quad \ldots (2)$$

From (2), we get $a^3 = -1$, i.e., $a = -1$ (considering only real roots)

Substituting $a = -1$ in (1), we have

$$-\frac{1}{r} - 1 - r = \frac{13}{12} \text{ or } 12r^2 + 25r + 12 = 0.$$

This is a quadratic in r, solving, we get $r = -\frac{3}{4}$ or $-\frac{4}{3}$.

Example15 Find the sum of the sequence 7, 77, 777, 7777, ... to n terms.
Solution This is not a G.P., however, we can relate it to a G.P. by writing the terms as

S_n = 7 + 77 + 777 + 7777 + ... to n terms

$= \frac{7}{9}$ [9+99+ 999+ 9999…. to n term]

$= \frac{7}{9}[(10-1)+ (10-1)^2+ (10-1)^3+ (10-1)^4+\ldots$ n terms]

$= \frac{7}{9}[(10+10^2+10^3+\ldots$ n terms) – (1+1+1…. n terms)]

$$= \frac{7}{9}\left[\frac{10(10^n-1)}{10-1}-n\right] = \frac{7}{9}\left[\frac{10(10^n-1)}{9}-n\right].$$

Example 16 A person has 2 parents, 4 grandparents, 8 great grandparents, and so on.
Find the number of his ancestors during the ten generations preceding his own.
Solution Here $a = 2$, $r = 2$ and $n = 10$

Using the sum formula $S_n = \frac{a(r^n-1)}{r-1}$

We have $S_{10} = 2(2^{10} - 1) = 2046$

Hence, the number of ancestors preceding the person is 2046.

Example17 Insert three numbers between 1 and 256 so that the resulting sequence
is a G.P.

Solution Let G1, G2,G3 be three numbers between 1 and 256 such that
1, G_1, G_2, G_3 ,256 is a G.P.

Therefore $256 = r^4$ giving $r = \pm 4$ (Taking real roots only)

For $r = 4$, we have $G_1 = ar = 4$, $G_2 = ar^2 = 16$, $G_3 = ar^3 = 64$

Similarly, for $r = -4$, numbers are – 4,16 and – 64.

Hence, we can insert 4, 16, 64 between 1 and 256 so that the resulting sequences are in G.P.

PYQ & EXPECTED QUESTIONS

Q) Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.

Q) The 4th term of a G.P. is square of its second term, and the first term is – 3. Determine its 7th term

Q) For what values of x, the numbers $\frac{-2}{7}, x, \frac{-7}{2}$ are in G.P.?

Q) How many terms of G.P. $3, 3^2, 3^3, \ldots$ are needed to give the sum 120?

Q) Find the sum to n terms of the sequence, 8, 88, 888, 8888… .

Q) Geometric mean of 16 and 4 is ……..

i) 20 ii) 4 iii) 10 iv) 8

Q) Find the 10th term of a G.P., whose 3rd term is 24 and 6th term is 192.