Chapter : 14
OSCILLATIONS

PERIODIC MOTION

- A motion that repeats at regular intervals of time.

Examples:
- Spinning of earth about its own axis
- Revolution of earth around the sun
- Oscillations of a pendulum
- Vibrations of a tuning fork

Types of periodic motion

Rotatory motion:-particle completes the rotation in regular intervals.
- Examples :-
- Rotation of earth around the sun
- Rotatory motion of hour hand, minute hand etc.

Oscillatory motion :- the particle moves to and fro with less frequency.
- **Examples** :- Bob of a pendulum
- Movement of swing

Vibratory motion:- the particle moves to and fro with large frequency.
- **Examples :-**
- The particle on a vibrating string
- Vibrations of atoms in a solid

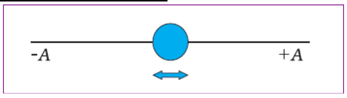
Period (T)
- The time taken to repeat a periodic motion is called the **period (T).**
- Its SI unit is second.
- The period of vibrations of a quartz crystal is expressed in units of microseconds (10^{-6} s) abbreviated as µs.
- The orbital period of the planet Mercury is 88 earth days.
- The Halley's Comet appears after every 76 years.

Frequency (ν)
- The number of repetitions in one second of a periodic motion is called **Frequency (ν).**
- Its unit is Hertz (Hz).
- The relation between *ν and T is,* $\nu = \frac{1}{T}$

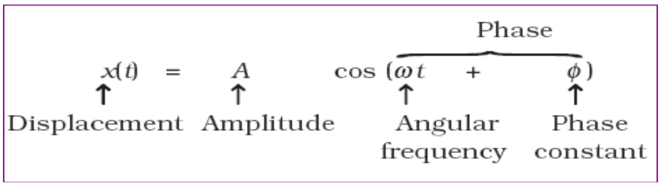
Relation connecting period (T) , angular velocity (ω) and Frequency (ν)
- The angular velocity, $\omega = 2\pi\nu$
- Or $\nu = \frac{\omega}{2\pi}$
- The period **T = (1/ ν) = 2 π/ ω**

SIMPLE HARMONIC MOTION (SHM)
- In SHM the restoring force ***on the oscillating body is directly proportional to its displacement from the mean position, and is directed opposite to the displacement.***
- Eg: small oscillations of simple pendulum, swing, loaded spring, etc.

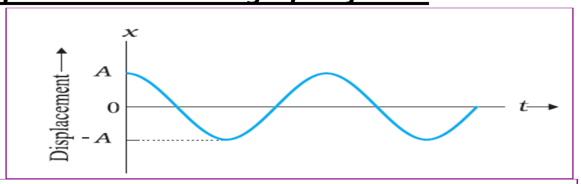
DISPLACEMENT OF SHM

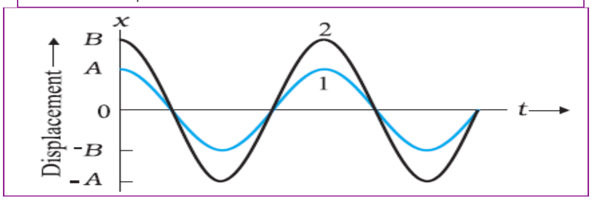


- The displacement is given by
$$x(t) = A\cos(\omega t + \phi)$$



Displacement – Time graph of SHM

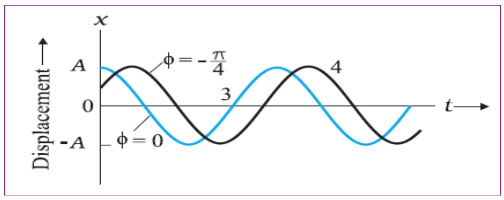




Amplitude(A):
- It is the magnitude of the maximum displacement of the oscillating particle

Phase:
- The time varying quantity, (ω*t + φ),* is called the **phase of the motion.**
- Phase describes the **state of motion** at a given time.

Phase constant (or phase angle):
- The constant ϕ is called the **phase constant.**
- The value of ϕ depends on the displacement and velocity of the particle at *t = 0.*

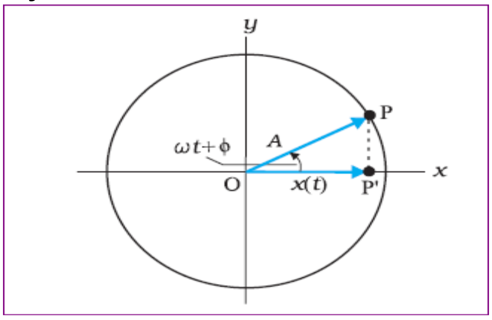


Angular frequency (ω):

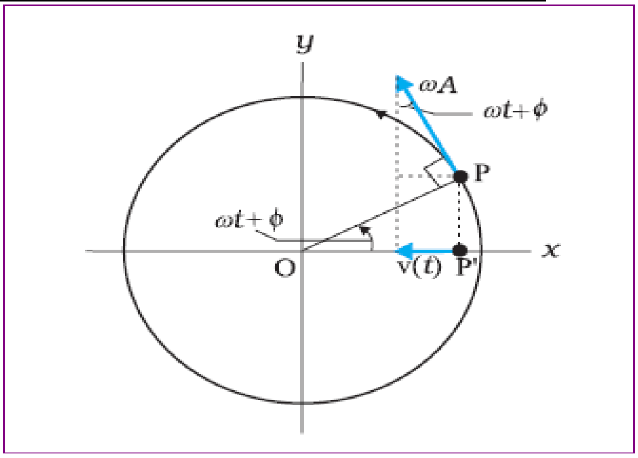
- The angular frequency is, $\omega = \dfrac{2\pi}{T}$
- The SI unit of angular frequency is ***radians per second***.

SIMPLE HARMONIC MOTION AND UNIFORM CIRCULAR MOTION

- ***Simple harmonic motion is the projection of uniform circular motion on a diameter of the circle.***



VELOCITY IN SIMPLE HARMONIC MOTION



- The displacement of SHM is given by
  $x(t) = A\cos(\omega t + \phi)$
- Differentiating with respect to time , we get the velocity
  $v(t) = -\omega A \sin(\omega t + \phi)$
- We know

$$\sin\ (\omega t + \phi) = \sqrt{1 - \cos^2(\omega t + \phi)}$$

- Thus $\quad v(t) = -\omega A\sqrt{1 - \cos^2(\omega t + \phi)}$

- Or $\quad v(t) = -\omega\sqrt{A^2 - x^2(t)}$

Case I

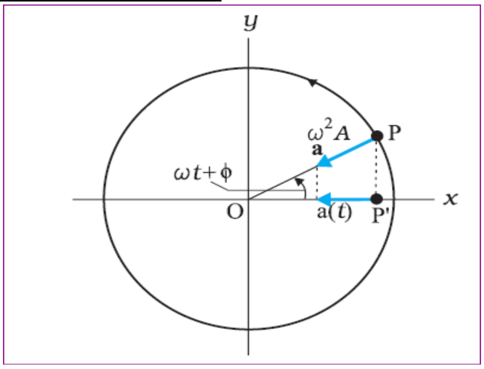
- **At mean position, x=0**
- Then magnitude of velocity

$$v_{\max} = \omega A$$

Case II

- **At extreme positions,** $x = \pm A$
- Then $v_{\min} = 0$

ACCELERATION IN SHM



- A particle executing a uniform circular motion is subjected to a radial acceleration **a directed towards the centre.**
- Differentiating the velocity we get the acceleration as

$$a(t) = -\omega^2 A\cos(\omega t + \phi)$$

- Or $\quad a(t) = -\omega^2 x(t)$
- Thus in **SHM, the acceleration is proportional to the displacement and is always directed towards the mean position.**

Case I

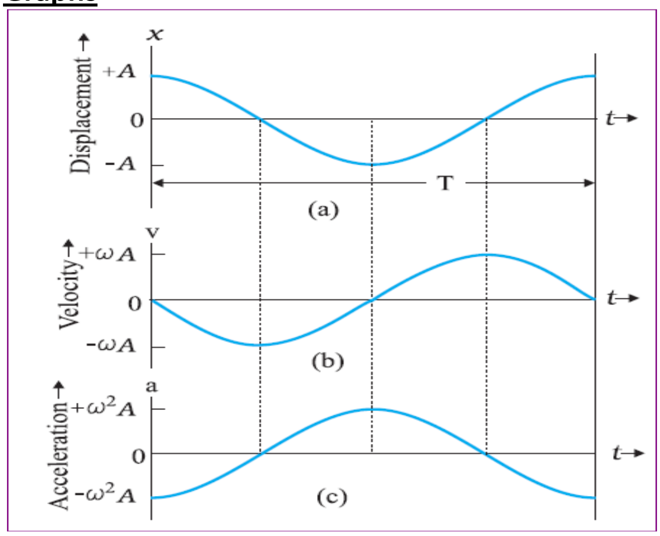
- **At mean position , x=0**
- Thus acceleration $a_{\min} = o$

Case II

- **At the extreme positions** , $x = \pm A$

$$a_{\max} = -A\omega^2$$

Graphs



FORCE LAW FOR SIMPLE HARMONIC MOTION

- The force law of SHM is
  $F(t) = -kx(t)$,
- where $k = m\omega^2$ is the force constant.

Derivation

- From Newton's law $F(t) = ma(t)$
- That is $F(t) = -m\omega^2 x(t)$
- Thus $F(t) = -kx(t)$
- $k = m\omega^2$, therefore

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

Linear harmonic oscillator

- The oscillator for which **restoring force is a linear function** of x.

***Non-linear* harmonic** or **anharmonic oscillators**

- Oscillators in which the **restoring force is a nonlinear function of *x*.**

DIFFERENTIAL EQUATION OF SHM

- The restoring force acting on a particle in SHM is given by , F = -kx
- From Newton's law we have F=ma
- But $a = \frac{d^2x}{dt^2}$
- Thus

$$m\frac{d^2x}{dt^2} = -kx$$

- Or

$$\frac{d^2x}{dt^2} + \frac{kx}{m} = 0$$

- That is

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

- Where $\omega^2 = \frac{k}{m}$
- For an angular displacement θ, the differential equation is,

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

ENERGY IN SIMPLE HARMONIC MOTION

- A particle executing simple harmonic motion has kinetic and potential energies

Kinetic Energy (K)

- The kinetic energy (*K) is*

$$K = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$$

Derivation

- We have, the kinetic energy $K = \frac{1}{2}mv^2$
- Substituting for velocity we get

$$K = \frac{1}{2}m\omega^2\sin^2(\omega t + \phi)$$

- That is $K = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$
- Thus kinetic energy is a periodic function of time.
- Kinetic energy is **<u>zero when the displacement is maximum</u>** and **<u>maximum at the mean position.</u>**
- The period of kinetic energy is **T/2**.

Potential Energy(PE)

- The potential energy of SHM is

$$U = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

Derivation

- The potential energy is given by $U = \frac{1}{2}kx^2$
- Thus $U = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$
- The potential energy of a particle executing simple harmonic motion is periodic, with period *T/2.*
- The **<u>potential energy is *zero at the mean* position</u>** and **<u>maximum at the extreme displacements.</u>**

Total Energy(E)

- The total energy, *E, of SHM is ,* $E = \frac{1}{2}kA^2$

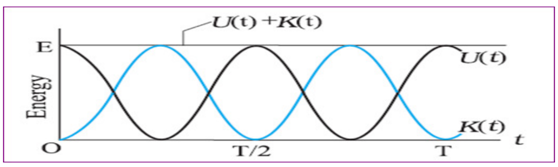
Derivation

- We have the the total energy $E = K + U$
  - Thus

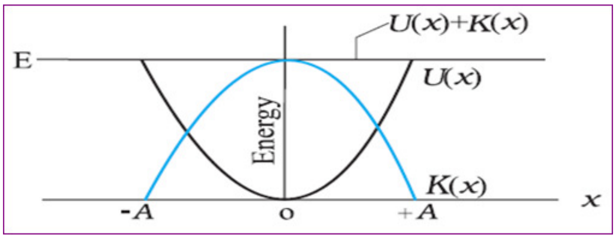
$$E = \frac{1}{2}kA^2\left[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right] = \frac{1}{2}kA^2$$

- The total mechanical energy of a harmonic oscillator is a constant.

Energy –Time graph of SHM

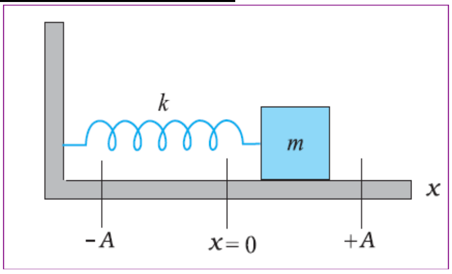


Energy –Displacement graph of SHM



EXAMPLES OF SIMPLE HARMONIC MOTION

Oscillations due to a Spring



- If the block is pulled on one side and is released, it then executes a to and fro motion about a mean position.
- At any time *t, the restoring force F* acting on the block is,

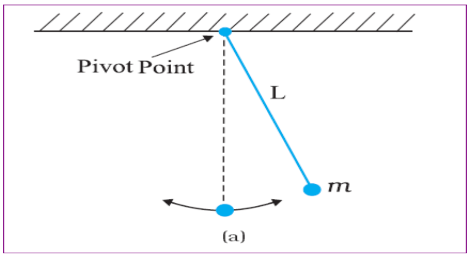
$$\boxed{F(x) = -kx}$$

- The constant of proportionality, *k, is called* the spring constant.
- A **stiff spring has large *k*** *and a* **soft spring has small k.**
- The equation is same as the force law for SHM and therefore the system executes a simple harmonic motion.
- The frequency of oscillations is $\omega = \sqrt{\frac{k}{m}}$
- The period is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

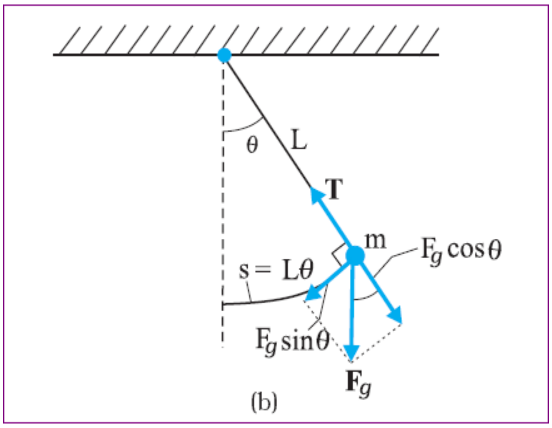
The Simple Pendulum

- A **simple pendulum,** consists of a particle of mass *m (called* the **bob** of the pendulum) suspended from one end of an unstretchable, massless string of length *L fixed at the other end.*



- **The forces acting on the bob are**:
  - Tension in the string
  - Gravitational force.

Expression for time period



- The string makes an angle θ with the vertical.
- We resolve the force $\mathbf{F_g}$ ***into a radial component*** $\mathbf{F_g \cos\theta}$ and a tangential component Fg sin θ.
- *The radial* component of force $\mathbf{F_g \cos\theta}$**,** is **cancelled by the tension**.
- The tangential component, $\mathbf{F_g \sin\theta}$ **produces a restoring torque .**
- The restoring torque τ is $\tau = -LF_g \sin\theta$
- Where the negative sign indicates that the torque acts to reduce θ.
- For rotational motion we have $\tau = I\alpha$
- where **I is the pendulum's moment of inertia** about the pivot point and **α is its angular acceleration** about that point.
- Thus , $-LF_g \sin\theta = I\alpha$
- But $F_g = mg$,
- Thus $-Lmg\sin\theta = I\alpha$
- Or

$$\alpha = -\frac{mgL}{I}\sin\theta$$

- If θ is small $\sin\theta \approx \theta$, therefore

$$\alpha = -\frac{mgL}{I}\theta$$

- That is, **the angular acceleration of the pendulum is proportional to the angular displacement θ but opposite in sign**.
- Thus the motion of a **simple pendulum swinging through small angles is approximately SHM.**

- Comparing equations $a(t) = -\omega^2 x(t)$ and $\alpha = -\frac{mgL}{I}\theta$, we get
- The angular frequency

$$\boxed{\omega = \sqrt{\frac{mgL}{I}}}$$

- And Period

$$T = 2\pi\sqrt{\frac{I}{mgL}}$$

- We have $I = mL^2$
- Thus

$$T = 2\pi\sqrt{\frac{L}{g}}$$

DAMPED SIMPLE HARMONIC MOTION

- The SHM which dies out due to the dissipative forces acting on it is called damped simple harmonic oscillation.
- Eg: free oscillations of a simple pendulum.

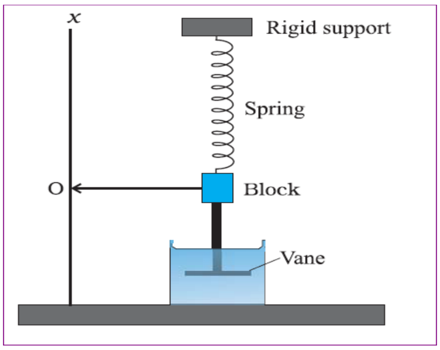
Differential equation of Damped oscillations

- Damped SHM is given by the equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

- Where m- mass, b –damping constant, k- spring constant

Derivation



- The damping force (viscous force) exerted by the liquid on the system is

$$F_d = -bv$$

- where *b is a **damping constant .***
- The negative sign shows that the force is opposite to the velocity.
- The restoring force on the spring is

$$F_S = -kx$$

- Thus the total force acting on the mass at any time t is

$$F = -kx - bv$$

- If **a** is the acceleration of the mass at time *t, then* by Newton's second law of motion

$$ma = -kx - bv$$

- Or

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Solution of the differential equation

- The solution of this equation is of the form

$$\boxed{x(t) = A e^{-bt/2m} \cos(\omega' t + \phi)}$$

- Where

$$\boxed{\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

Amplitude of damped oscillator

- The amplitude is given by

$$Amplitude = Ae^{-bt/2m}$$

- Thus **amplitude decreases with time**.

Undamped oscillator

- *If b=0*(there is no damping), then

$$x(t) = A\cos(\omega' t + \phi)$$

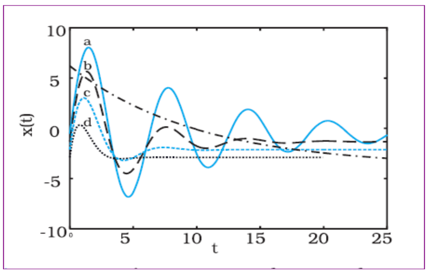
$$\omega' = \sqrt{\frac{k}{m}}$$

Energy of a damped oscillator

- If the oscillator is damped, the mechanical energy decreases with time.
- The total energy is given by

$$\boxed{E(t) = \frac{1}{2} k A^2 e^{-bt/m}}$$

Displacement – time graph of damped oscillator



Free Oscillations

- The oscillations of a body oscillated and left free, are called ***free oscillations***.
- The frequency of free oscillation is known as ***natural frequency.***

- A person swinging in a swing without anyone pushing it or a simple pendulum, displaced and released, are examples of free oscillations.

FORCED OSCILLATIONS

- When a periodic force is applied to maintain an oscillation, it is known as ***forced*** or ***driven oscillations***.
- The frequency of forced oscillations is equal to the frequency of the applied force.
- The two angular frequencies in a forced oscillation are
    - **The natural angular frequency ω of the system**
    - **The angular frequency ω_d** *of the external force* causing the driven oscillations.
- The oscillator initially oscillates with its natural frequency ω.
- When we apply the external periodic force, the oscillations with the natural frequency die out, and then the body oscillates with the (angular) frequency of the external periodic force.

Differential equation of forced oscillation

- The forced oscillation is given by

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\cos\omega_d t$$

- Where F_0 – amplitude of periodic force.
- The periodic force is

$$\boxed{F(t) = F_o \cos \omega_d t}$$

Derivation

- The motion of a particle under the combined action of a linear restoring force, damping force and a time dependent driving force is given by

$$\boxed{m\,a(t) = -k\,x(t) - bv(t) + F_o \cos \omega_d t}$$

- Or

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\cos\omega_d t$$

Solution of the Differential equation

- The solution of the equation is

$$\boxed{x(t) = A\cos(\omega_d t + \phi)}$$

- Where

$$A=\frac{F_0}{\left\{m^2\left(\omega^2-\omega_d^2\right)^2+\omega_d^2b^2\right\}^{1/2}}$$

- And

$$\boxed{\tan\phi = \frac{-v_o}{\omega_d x_o}}$$

- Where *m - mass of the particle, v_0 and x_0 are the velocity and the displacement of the* particle at time *t = 0.*

Special Cases

1. **Small Damping, Driving Frequency far from Natural Frequency :**

- In this case, **$\omega_d b$** *will* be much smaller than *$m(\omega^2-\omega^2 d)$ and we can* neglect that term.

$$\boxed{A=\frac{F_0}{\left\{m\ \left(\omega^2-\omega_d^{\,2}\right)\right\}}}$$

- If we go on changing the driving frequency, the amplitude tends to infinity when it equals the natural frequency

2. ***Driving Frequency Close to Natural Frequency***

- If ω_d *is very close to ω ,then* ***$m(\omega^2-\omega^2 d)$*** would be much less than ***$\omega_d b$.***
- Therefore

$$\boxed{A=\frac{F_o}{\omega_d b}}$$

- Thus the maximum possible amplitude for a given driving frequency is governed by the driving frequency and the damping.

Resonance

- The phenomenon of increase in amplitude when the ***driving force is close to the natural frequency*** of the oscillator is called ***resonance.***
- The Tacoma Narrows Bridge, USA collapsed due to resonance..
- To avoid rupture of bridges due to resonance , ***marching soldiers break steps while crossing a bridge***.
- ***Aircraft designer***s make sure that none of the natural frequencies at which a wing can oscillate match the frequency of the engines in flight.
- In an ***earthquake, short and tall structures remain unaffected while the medium height structures fall down.***
- The natural frequencies of the short structures are higher and those of taller structures lower than the frequency of the seismic waves.

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