JAM 2006

MATHEMATICAL STATISTICS TEST PAPER

Special Instructions / Useful Data

1. For an event A, $P(A)$ denotes the probability of the event A.
2. The complement of an event is denoted by putting a superscript "c" on the event, e.g. A^c denotes the complement of the event A.
3. For a random variable X, $E(X)$ denotes the expectation of X and $V(X)$ denotes its variance.
4. $N(\mu, \sigma^2)$ denotes a normal distribution with mean μ and variance σ^2.
5. Standard normal random variable is a random variable having a normal distribution with mean 0 and variance 1.
6. $P(Z > 1.96) = 0.025$, $P(Z > 1.65) = 0.050$, $P(Z > 0.675) = 0.250$ and $P(Z > 2.33) = 0.010$, where Z is a standard normal random variable.
7. $P(\chi_2^2 \geq 9.21) = 0.01$, $P(\chi_2^2 \geq 0.02) = 0.99$, $P(\chi_3^2 \geq 11.34) = 0.01$, $P(\chi_4^2 \geq 9.49) = 0.05$, $P(\chi_4^2 \geq 0.71) = 0.95$, $P(\chi_5^2 \geq 11.07) = 0.05$ and $P(\chi_5^2 \geq 1.15) = 0.95$, where $P(\chi_n^2 \geq c) = \alpha$, where χ_n^2 has a Chi-square distribution with n degrees of freedom.
8. $n!$ denotes the factorial of n.
9. The determinant of a square matrix A is denoted by $|A|$.
10. R: The set of all real numbers.
11. R": n-dimensional Euclidean space.
12. y' and y'' denote the first and second derivatives respectively of the function $y(x)$ with respect to x.

NOTE: This Question-cum-Answer book contains THREE sections, the Compulsory Section A, and the Optional Sections B and C.

- **Attempt ALL questions in the compulsory section A. It has 15 objective type questions of *six* marks each and also *nine* subjective type questions of *fifteen* marks each.**
- **Optional Sections B and C have *five* subjective type questions of *fifteen* marks each.**
- **Candidates seeking admission to either of the two programmes, M.Sc. in Applied Statistics & Informatics at IIT Bombay and M.Sc. in Statistics & Informatics at IIT Kharagpur, are required to attempt ONLY Section B (Mathematics) from the Optional Sections.**
- **Candidates seeking admission to the programme, M.Sc. in Statistics at IIT Kanpur, are required to attempt ONLY Section C (Statistics) from the Optional Sections.**

You must therefore attempt either Optional Section B or Optional Section C depending upon the programme(s) you are seeking admission to, and accordingly tick one of the boxes given below.

Optional Section Attempted	**B**	
	C	

- ***The negative marks for the Objective type questions will be carried over to the total marks*.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page MS 11/63 only.**

Compulsory Section A

1. If $a_n > 0$ for $n \geq 1$ and $\lim_{n \to \infty} \left(a_n\right)^{1/n} = L < 1$, then which of the following series is not convergent?

   (A) $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$

   (B) $\sum_{n=1}^{\infty} a_n^2$

   (C) $\sum_{n=1}^{\infty} \sqrt{a_n}$

   (D) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}}$

2. Let E and F be two mutually disjoint events. Further, let E and F be independent of G. If $p = P(E) + P(F)$ and $q = P(G)$, then $P(E \cup F \cup G)$ is

   (A) $1 - pq$

   (B) $q + p^2$

   (C) $p + q^2$

   (D) $p + q - pq$

3. Let X be a continuous random variable with the probability density function symmetric about 0. If $V\left(X\right) < \infty$, then which of the following statements is true?

   (A) $E\left(|X|\right) = E\left(X\right)$

   (B) $V\left(|X|\right) = V\left(X\right)$

   (C) $V\left(|X|\right) < V\left(X\right)$

   (D) $V\left(|X|\right) > V\left(X\right)$

4. Let

   $$f(x) = x|x| + |x-1|, \quad -\infty < x < \infty.$$

   Which of the following statements is true?

   (A) f is not differentiable at $x = 0$ and $x = 1$.

   (B) f is differentiable at $x = 0$ but not differentiable at $x = 1$.

   (C) f is not differentiable at $x = 0$ but differentiable at $x = 1$.

   (D) f is differentiable at $x = 0$ and $x = 1$.

5. Let $A\underset{\sim}{x} = \underset{\sim}{b}$ be a non-homogeneous system of linear equations. The augmented matrix $\left[A \,:\, \underset{\sim}{b}\right]$ is given by

   $$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ -1 & 2 & 3 & -1 & 0 \\ 0 & 3 & 1 & 0 & -1 \end{array}\right].$$

Which of the following statements is true?

(A) Rank of A is 3.
(B) The system has no solution.
(C) The system has unique solution.
(D) The system has infinite number of solutions.

6. An archer makes 10 independent attempts at a target and his probability of hitting the target at each attempt is $\frac{5}{6}$. Then the conditional probability that his last two attempts are successful given that he has a total of 7 successful attempts is

(A) $\frac{1}{5^5}$
(B) $\frac{7}{15}$
(C) $\frac{25}{36}$
(D) $\frac{8!}{3!\ 5!}\left(\frac{5}{6}\right)^7\left(\frac{1}{6}\right)^3$

7. Let

$$f(x)=(x-1)(x-2)(x-3)(x-4)(x-5),\quad -\infty < x < \infty.$$

The number of distinct real roots of the equation $\frac{d}{dx}f(x)=0$ is exactly

(A) 2 (B) 3 (C) 4 (D) 5

8. Let

$$f(x)=\frac{k\ |x|}{(1+|x|)^4},\quad -\infty < x < \infty.$$

Then the value of k for which $f(x)$ is a probability density function is

(A) $\frac{1}{6}$
(B) $\frac{1}{2}$
(C) 3
(D) 6

9. If $M_X(t) = e^{3t+8t^2}$ is the moment generating function of a random variable X, then $P(-4.84 < X \leq 9.60)$ is

(A) equal to 0.700
(B) equal to 0.925
(C) equal to 0.975
(D) greater than 0.999

10. Let X be a binomial random variable with parameters n and p, where n is a positive integer and $0 \le p \le 1$. If $\alpha = P\left(|X - np| \ge \sqrt{n}\right)$, then which of the following statements holds true for all n and p?

(A) $0 \le \alpha \le \frac{1}{4}$

(B) $\frac{1}{4} < \alpha \le \frac{1}{2}$

(C) $\frac{1}{2} < \alpha < \frac{3}{4}$

(D) $\frac{3}{4} \le \alpha \le 1$

11. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter p; $0 \le p \le 1$. The bias of the estimator $\dfrac{\sqrt{n} + 2\sum_{i=1}^{n} X_i}{2\left(n + \sqrt{n}\right)}$ for estimating p is equal to

(A) $\dfrac{1}{\sqrt{n}+1}\left(p - \dfrac{1}{2}\right)$

(B) $\dfrac{1}{n+\sqrt{n}}\left(\dfrac{1}{2} - p\right)$

(C) $\dfrac{1}{\sqrt{n}+1}\left(\dfrac{1}{2} + \dfrac{p}{\sqrt{n}}\right) - p$

(D) $\dfrac{1}{\sqrt{n}+1}\left(\dfrac{1}{2} - p\right)$

12. Let the joint probability density function of X and Y be

$$f(x, y) = \begin{cases} e^{-x}, & \text{if } 0 \le y \le x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then $E(X)$ is

(A) 0.5

(B) 1

(C) 2

(D) 6

13. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(t) = \begin{cases} \dfrac{\tan t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Then the value of $\lim_{x \to 0} \frac{1}{x^2} \int_{x^2}^{x^3} f(t)\, dt$

(A) is equal to -1
(B) is equal to 0
(C) is equal to 1
(D) does not exist

14. Let X and Y have the joint probability mass function;

$$P(X = x, Y = y) = \frac{1}{2^{y+2}(y+1)} \left(\frac{2y+1}{2y+2} \right)^x, \quad x, y = 0, 1, 2, \ldots .$$

Then the marginal distribution of Y is

(A) Poisson with parameter $\lambda = \frac{1}{4}$

(B) Poisson with parameter $\lambda = \frac{1}{2}$

(C) Geometric with parameter $p = \frac{1}{4}$

(D) Geometric with parameter $p = \frac{1}{2}$

15. Let X_1, X_2 and X_3 be a random sample from a $N(3, 12)$ distribution. If $\bar{X} = \frac{1}{3} \sum_{i=1}^{3} X_i$ and $S^2 = \frac{1}{2} \sum_{i=1}^{3} (X_i - \bar{X})^2$ denote the sample mean and the sample variance respectively, then $P(1.65 < \bar{X} \leq 4.35,\ 0.12 < S^2 \leq 55.26)$ is

(A) 0.49
(B) 0.50
(C) 0.98
(D) none of the above

16. (a) Let $X_1, X_2, \dots, X_n$ be a random sample from an exponential distribution with the probability density function;

$$f(x;\theta) = \begin{cases} \theta\, e^{-\theta x}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$. Obtain the maximum likelihood estimator of $P(X > 10)$. **9 Marks**

(b) Let $X_1, X_2, \dots, X_n$ be a random sample from a discrete distribution with the probability mass function given by

$$P(X = 0) = \frac{1-\theta}{2}; \quad P(X = 1) = \frac{1}{2}; \quad P(X = 2) = \frac{\theta}{2}, \quad 0 \le \theta \le 1.$$

Find the method of moments estimator for θ. **6 Marks**

17. (a) Let A be a non-singular matrix of order n $(n > 1)$, with $|A| = k$. If $adj(A)$ denotes the adjoint of the matrix A, find the value of $|adj(A)|$. **6 Marks**

(b) Determine the values of a, b and c so that $(1,\ 0,\ -1)$ and $(0,\ 1,\ -1)$ are eigenvectors of the matrix,

$$\begin{bmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 3 & b & c \end{bmatrix}.$$ **9 Marks**

18. (a) Using Lagrange's mean value theorem, prove that

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2},$$

where $0 < \tan^{-1} a < \tan^{-1} b < \dfrac{\pi}{2}$. **6 Marks**

(b) Find the area of the region in the first quadrant that is bounded by $y = \sqrt{x}$, $y = x - 2$ and the $x-$axis. **9 Marks**

19. Let X and Y have the joint probability density function;

$$f(x,y) = \begin{cases} c\,x\,y\; e^{-\left(x^2 + 2y^2\right)}, & \text{if } x > 0, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the constant c and $P\left(X^2 > Y^2\right)$.

20. Let PQ be a line segment of length β and midpoint R. A point S is chosen at random on PQ. Let X, the distance from S to P, be a random variable having the uniform distribution on the interval $(0, \beta)$. Find the probability that PS, QS and PR form the sides of a triangle.

21. Let $X_1, X_2, \dots, X_n$ be a random sample from a $N(\mu, 1)$ distribution. For testing $H_0 : \mu = 10$ against $H_1 : \mu = 11$, the most powerful critical region is $\overline{X} \ge k$, where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find k in terms of n such that the size of this test is 0.05. Further determine the minimum sample size n so that the power of this test is at least 0.95.

22. Consider the sequence $\{s_n\}$, $n \geq 1$, of positive real numbers satisfying the recurrence relation

$$s_{n-1} + s_n = 2\, s_{n+1} \quad \text{for all} \quad n \geq 2\,.$$

(a) Show that $|\, s_{n+1} - s_n \,| = \dfrac{1}{2^{n-1}} |\, s_2 - s_1 \,|$ for all $n \geq 1$.

(b) Prove that $\{s_n\}$ is a convergent sequence.

23. The cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ \dfrac{1}{5}\left(1 + x^3\right), & \text{if } 0 \leq x < 1, \\ \dfrac{1}{5}\left[3 + \left(x-1\right)^2\right], & \text{if } 1 \leq x < 2, \\ 1, & \text{if } x \geq 2. \end{cases}$$

Find $P\left(0 < X < 2\right)$, $P\left(0 \leq X \leq 1\right)$ and $P\left(\dfrac{1}{2} \leq X \leq \dfrac{3}{2}\right)$.

24. Let A and B be two events with $P\left(A \,|\, B\right) = 0.3$ and $P\left(A \,|\, B^c\right) = 0.4$. Find $P(B \,|\, A)$ and $P(B^c \,|\, A^c)$ in terms of $P(B)$. If $\dfrac{1}{4} \leq P(B \,|\, A) \leq \dfrac{1}{3}$ and $\dfrac{1}{4} \leq P(B^c \,|\, A^c) \leq \dfrac{9}{16}$, then determine the value of $P(B)$.

Optional Section B

25. Solve the initial value problem

$$y' - y + y^2 \left(x^2 + 2\,x + 1\right) = 0, \;\; y(0) = 1.$$

26. Let $y_1(x)$ and $y_2(x)$ be the linearly independent solutions of

$$x\, y'' + 2\, y' + x\, e^x\, y = 0.$$

If $W(x) = y_1(x)\; y_2'(x) - y_2(x)\; y_1'(x)$ with $W(1) = 2$, find $W(5)$.

27. (a) Evaluate $\displaystyle\int_0^1 \int_y^1 x^2\, e^{xy}\, dx\, dy.$ **9 Marks**

(b) Evaluate $\displaystyle\iiint_W z \;\, dx\, dy\, dz$, where W is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = 1$ and the cylinder $x^2 + y^2 = 1$ with $x \geq 0, y \geq 0$.

6 Marks

28. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$T\left(x, y, z\right) = \left(3\,x + 11\,y + 5\,z,\;\; x + 8\,y + 3\,z\right).$$

Determine the matrix representation of this transformation relative to the ordered bases $\left\{(1, 0, 1), (0, 1, 1), (1, 0, 0)\right\}$, $\left\{(1, 1), (1, 0)\right\}$. Also find the dimension of the null space of this transformation.

29. (a) Let $f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{x + y}, & \text{if } x + y \neq 0, \\ 0, & \text{if } x + y = 0. \end{cases}$

Determine if f is continuous at the point $(0, 0)$. **6 Marks**

(b) Find the minimum distance from the point $(1, 2, 0)$ to the cone $z^2 = x^2 + y^2$. **9 Marks**

Optional Section C

30. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with the probability density function;

$$f(x; \theta) = \begin{cases} \dfrac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$. Derive the Cramér-Rao lower bound for the variance of any unbiased estimator of θ. Hence, prove that $T = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ is the uniformly minimum variance unbiased estimator of θ.

31. Let $X_1, X_2, \ldots$ be a sequence of independently and identically distributed random variables with the probability density function;

$$f(x) = \begin{cases} \dfrac{1}{2} x^2 e^{-x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\lim\limits_{n \to \infty} P\left(X_1 + \ldots + X_n \geq 3\left(n - \sqrt{n}\right)\right) \geq \dfrac{1}{2}$.

32. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N\left(\mu, \sigma^2\right)$ distribution, where both μ and σ^2 are unknown. Find the value of b that minimizes the mean squared error of the estimator $T_b = \dfrac{b}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ for estimating σ^2, where $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$.

33. Let $X_1, X_2, \ldots, X_5$ be a random sample from a $N\left(2, \sigma^2\right)$ distribution, where σ^2 is unknown. Derive the most powerful test of size $\alpha = 0.05$ for testing $H_0 : \sigma^2 = 4$ against $H_1 : \sigma^2 = 1$.

34. Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with the probability density function;

$$f(x; \lambda) = \begin{cases} \dfrac{2x}{\lambda} e^{-\frac{x^2}{\lambda}}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$. Find the maximum likelihood estimator of λ and show that it is sufficient and an unbiased estimator of λ.