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2008-MS

Test Paper Code: MS

Time: 3 Hours **Maximum Marks: 300**

INSTRUCTIONS

1. The question-cum-answer booklet has **28** pages and has **25** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Roll Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **5**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used:
   (a) For each correct answer, you will be awarded **6 (Six)** marks.
   (b) For each wrong answer, you will be awarded **-2 (Negative two)** mark.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, calculator, cellular phone, pager and electronic gadgets in any form are NOT allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on the reverse.

A

2008-MS

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

ROLL NUMBER						

Name:

Test Centre:

Do not write your Roll Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

....

Signature of the Candidate

I have verified the information filled by the Candidate above.

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Signature of the Invigilator

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Special Instructions/ Useful Data

1. $[x]$: Greatest integer in x.
2. For an event A, $P(A)$ denotes the probability of the event A.
3. For a random variable X, $E(X)$ denotes the expectation of X.
4. Poisson(λ): Poisson distribution with parameter λ.
5. Binomial(n, p): Binomial distribution with parameters n and p.
6. χ^2_m: Chi-square distribution with m degrees of freedom.
7. Exp(θ): Exponential distribution with mean θ.
8. $N(\mu, \sigma^2)$: Normal distribution with mean μ and variance σ^2.
9. $U(a, b)$: Uniform distribution on the interval $[a, b]$.
10. $\bar{X} = \dfrac{X_1 + X_2 + ... + X_n}{n}$.
11. $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$.
12. $X_{(n)} = \max\{X_1, X_2, ..., X_n\}$.
13. Corr(X, Y): Correlation coefficient between X and Y.
14. $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$.
15. $n! = n(n-1)...1$.
16. $\binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$
17. $\mathbb{R}$: The set of all real numbers.
18. $f'(x), f''(x), f'''(x)$ and $f^{iv}(x)$ denote the first, second, third and the fourth derivatives, respectively, of the function $f(x)$ with respect to x.
19. $y'(0) = \left.\dfrac{dy(x)}{dx}\right|_{x=0}$

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IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-25 (subjective questions) carry *twenty one* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 5 only.**

Q.1 Consider the following two series

$$S_1 = \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+3)}, \quad S_2 = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}\,\sqrt{k+3}}.$$

Then

(A) S_1 and S_2 converge (B) S_1 diverges and S_2 converges

(C) S_1 converges and S_2 diverges (D) S_1 and S_2 diverge

Q.2 Let $f(x) = [x]$ and

$$g(x) = \begin{cases} x, & 0 \le x < 1, \\ x-1, & 1 \le x < 2, \\ x-2, & 2 \le x < 3, \\ 0, & x = 3, \end{cases}$$

for $x \in [0,3]$. Then $f(x) + g(x)$ is

(A) discontinuous at points 1 and 2

(B) continuous on $[0,3]$ but not differentiable on $(0,3)$

(C) differentiable once but not twice on $(0,3)$

(D) twice differentiable on $(0,3)$

Q.3 The area of the region enclosed by the curve $y = x^2$ and the straight line $x + y = 2$ is

(A) 3 (B) $\frac{27}{2}$ (C) $\frac{9}{2}$ (D) 9

Q.4 If

$$\int_0^x f(t)\,dt = x^2 \sin x + x^3,$$

then $f\left(\frac{\pi}{2}\right)$ is

(A) $\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2}\right)^3$ (B) $\pi + \frac{3\pi^2}{4}$ (C) $\pi - \frac{3\pi^2}{4}$ (D) 0

Q.5 If A is a 3×3 non-zero matrix such that $A^2=0$, then the number of non-zero eigenvalues of A is

(A) 0 (B) 1 (C) 2 (D) 3

Q.6 The solution of the differential equation

$$\frac{dy}{dx}=-\frac{x\left(x^2+y^2-10\right)}{y\left(x^2+y^2+5\right)},\quad y(0)=1$$

is

(A) $x^4-2x^2y^2-y^4-20x^2-10y^2+11=0$

(B) $x^4+2x^2y^2+y^4+20x^2+10y^2-11=0$

(C) $x^4+2x^2y^2-y^4+20x^2-10y^2+11=0$

(D) $x^4+2x^2y^2+y^4-20x^2+10y^2-11=0$

Q.7 Let X be Poisson(2) and Y be Binomial$(10,3/4)$ random variables. If X and Y are independent, then $P(XY=0)$ is

(A) $e^{-2}+\left(\frac{1}{4}\right)^{10}\left(1-e^{-2}\right)$ (B) $e^{-2}+\left(\frac{1}{4}\right)^{10}\left(1-2e^{-2}\right)$

(C) $e^{-2}\left(\frac{1}{4}\right)^{10}$ (D) $e^{-2}+1-\left(\frac{1}{4}\right)^{10}$

Q.8 Let $X_1,X_2,\ldots,X_n$ $(n>1)$ be a random sample from Exp(1). Then the distribution of $\left(2n\bar{X}\right)$ is

(A) $\text{Exp}\left(\frac{1}{2}\right)$ (B) $\text{Exp}(2n)$ (C) χ_n^2 (D) χ_{2n}^2

Q.9 Suppose X is a random variable with finite variance. For $0<\theta<1$ and $n>3$, let $X_1=X$, $X_2=\theta X_1$, $X_3=\theta X_2,\ldots,X_n=\theta X_{n-1}$. Then $\text{Corr}(X_1,X_n)$ is

(A) 1 (B) -1 (C) θ^{n-2} (D) θ^{n-3}

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Q.10 Let X_1, X_2, X_3 be independent random variables with X_k $(k=1,2,3)$ having the probability density function

$$f_k(x)=\begin{cases} k\theta e^{-k\theta x}, & 0<x<\infty \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta>0$. Then a sufficient statistic for θ is

(A) $X_1+X_2+X_3$ (B) $3X_1+2X_2+X_3$

(C) $X_1+X_2+3X_3$ (D) $X_1+2X_2+3X_3$

Q.11 Let X be a random variable such that $E|X|<\infty$ and

$$P\left(X\geq\frac{1}{2}+x\right)=P\left(X\leq\frac{1}{2}-x\right)$$

for all $x\in\mathbb{R}$. Then

(A) $\mathrm{E}(X)=\frac{1}{2}$ and $\text{Median}(X)=\frac{1}{2}$ (B) $\mathrm{E}(X)=\frac{1}{2}$ and $\text{Median}(X)>\frac{1}{2}$

(C) $\mathrm{E}(X)<\frac{1}{2}$ and $\text{Median}(X)=\frac{1}{2}$ (D) $\mathrm{E}(X)<\frac{1}{2}$ and $\text{Median}(X)>\frac{1}{2}$

Q.12 Let $X_1, X_2, \ldots$ be a sequence of independent random variables. Suppose, for $k=1,2,\ldots,$

$$P\left(X_{2k-1}=1\right)=P\left(X_{2k-1}=-1\right)=\frac{1}{2}$$

and the probability density function of X_{2k} is

$$f(x)=\frac{1}{\sqrt{2}}e^{-\sqrt{2}|x|}, \quad -\infty<x<\infty.$$

Then $\lim_{n\to\infty} P\left[\frac{X_1+X_2+\ldots+X_{2n}}{\sqrt{2n}}\geq 1\right]$ is

(A) $\Phi(1)$ (B) $\frac{1}{2}$ (C) $\Phi(-1)$ (D) 1

Q.13 Let $X_1, X_2, \ldots, X_9$ be a random sample from $N(\theta,1)$, where $-\infty<\theta<\infty$. Consider the following two tests for testing $H_0:\theta=2.5$ against $H_1:\theta=4$

Test 1: Reject H_0 if $X_1>4$,

Test 2: Reject H_0 if $\bar{X}>3$.

Suppose α_k and β_k are the probabilities of Type I and Type II errors, respectively, for Test k $(k=1,2)$. Then

(A) $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$ (B) $\alpha_1=\alpha_2$ and $\beta_1>\beta_2$

(C) $\alpha_1>\alpha_2$ and $\beta_1=\beta_2$ (D) $\alpha_1=\alpha_2$ and $\beta_1<\beta_2$

Q.14 Let $X_1, X_2, \ldots, X_n$ $(n>1)$ be a random sample from $U(\theta, \theta+1)$. Consider the following three estimators for θ, $\theta \in \mathbb{R}$

$$T_1 = X_{(n)},$$
$$T_2 = \frac{X_{(1)} + X_{(n)}}{2},$$
$$T_3 = \frac{X_{(1)} + X_{(n)}}{2} - \frac{1}{2}.$$

Then

(A) T_1 and T_3 both are maximum likelihood estimators of θ while T_2 is not
(B) T_3 is the unique maximum likelihood estimator of θ
(C) T_3 is a maximum likelihood estimator of θ while T_1 and T_2 are not
(D) T_1 and T_2 both are maximum likelihood estimators of θ while T_3 is not

Q.15 A nonempty subset P is formed by selecting elements at random and without replacement from a set B consisting of $n\,(>1)$ distinct elements. Another nonempty subset Q is formed in a similar fashion from the original set B consisting of the same n elements. Then the probability that P and Q do not have any common element is

(A) $\dfrac{\sum_{i=1}^{n}\sum_{j=1}^{n-i}\binom{n}{i}\binom{n-i}{j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}\binom{n}{i}\binom{n}{j}}$

(B) $\dfrac{\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}}{\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}}$

(C) $\dfrac{\sum_{i=1}^{n-1}\sum_{j=1}^{n-i}\binom{n}{i}\binom{n-i}{j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}\binom{n}{i}\binom{n}{j}}$

(D) $\dfrac{\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}}{\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}}$

Space for rough work

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Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question No. Do not write anything else on this page.

Question No.	Answer	Do not write in this column
01		
02		
03		
04		
05		
06		
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09		
10		
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12		
13		
14		
15		

FOR EVALUATION ONLY

No. of Correct Answers		Marks	(+)
No. of Incorrect Answers		Marks	(−)
Total Marks in Question Nos. 1-15			( )

Q.16 (a) One coin is selected at random from two coins. The probability of obtaining head for one of them is $\frac{1}{3}$ and for the other it is $\frac{1}{2}$. If the selected coin is tossed and the head shows up, what is the probability that it is the fair coin? (9)

(b) Let p denote the probability that the weather (either wet or dry) tomorrow will be the same as that of today. If the weather is dry today, show that P_n, the probability that it will be dry n days later, satisfies

$$P_n = (2p-1)\,P_{n-1} + (1-p), \quad n \geq 1.$$

Hence, or otherwise, determine the value of P_{50} for $p = \frac{3}{4}$. (12)

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Q.17 (a) Let X and Y have the joint probability mass function

$$P(X=x,Y=y)=\begin{cases}\dfrac{e^{-2}}{x!\,(y-x)!}, & x=0,1,2,\ldots,y; \quad y=0,1,2,\ldots\\ 0, & \text{otherwise.}\end{cases}$$

Determine $M(t_1, t_2)$, the joint moment generating function of (X,Y). (9)

(b) The conditional probability density function of X given $Y=y\,(>0)$ is

$$f(x\mid y)=\begin{cases}ye^{-yx}, & x>0,\\ 0, & \text{otherwise,}\end{cases}$$

and the marginal probability density function of Y is

$$g(y;\alpha)=\begin{cases}\alpha e^{-\alpha y}, & y>0,\\ 0, & \text{otherwise.}\end{cases}$$

Derive the conditional probability density function of Y given $X=x$. (12)

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Q.18 Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution having the probability density function

$$f(x;\,\theta)=\begin{cases}\dfrac{2x}{\theta^2}, & 0<x\le\theta,\\ 0, & \text{otherwise.}\end{cases}$$

(a) Find the maximum likelihood estimator $\hat{\theta}$ of θ and determine the constant c such that $E\left(c\,\hat{\theta}\right)=\theta$. (12)

(b) Find the method of moments estimator of θ. (9)

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Q.19 Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution having the probability density function

$$f(x;\theta)=\begin{cases}\theta\, 2^{\theta}\, x^{-(\theta+1)}, & x>2,\\ 0, & \text{otherwise,}\end{cases}$$

where, $\theta > 2$.

(a) Show that $\sum_{k=1}^{n} \ln X_k$ is sufficient and complete for θ. (12)

(b) Find the Cramér-Rao lower bound of the variance of an unbiased estimator of $(\ln\theta)$. (9)

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Q.20 Let X_1 and X_2 be a random sample from a distribution having the probability density function $f(x)$. Consider the testing of $H_0 : f(x) = f_0(x)$ against $H_1 : f(x) = f_1(x)$ based on X_1 and X_2, where

$$f_0(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_1(x) = \begin{cases} 4x^3, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) For a given level, show that the critical region of the most powerful test for testing H_0 against H_1 is of the form

$$\{(x_1, x_2) : \ln x_1 + \ln x_2 > c\},$$

for some constant c. (9)

(b) Determine c in terms of a suitable cutoff point of a Chi-square distribution when the level is α. (12)

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Q.21 (a) The random variable X has the distribution function

$$F(x)=\begin{cases} 0, & x<0, \\ \dfrac{2x^2+1}{10}, & 0 \le x < 1, \\ \dfrac{4}{5}, & 1 \le x < 2, \\ \dfrac{(x-2)^4+16}{20}, & 2 \le x < 3, \\ 1, & x \ge 3. \end{cases}$$

Find the values of $P(1 \le X \le 2.5)$ and $P(0.5 < X < 3)$. (12)

(b) Let X be a normal random variable with mean 0 and variance 1. Show that

$$P(X \ge c) \le e^{-ct+\frac{t^2}{2}}$$

for $c>0$ and $t \in \mathbb{R}$. (9)

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Q.22 (a) Let $a_1 = 0$ and

$$a_{n+1} = \frac{a_n^2 + 3}{2(a_n + 1)} \quad \text{for } n \geq 1.$$

Show that the sequence $\{a_n\}$ converges and find its limit. (12)

(b) Let

$$a_{2n-1} = \frac{9^{n-1}}{16^{n-1}} \text{ and } a_{2n} = \frac{9^{n-1}}{16^n} \text{ for } n \geq 1.$$

Test whether $\sum_{n=1}^{\infty} a_n$ is convergent. (9)

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Q.23 (a) Let $f:[a,b]\to\mathbb{R}$ be a four times differentiable function such that $f^{iv}(x)>0$ for $x\in(a,b)$. If $c\in(a,b)$ is such that $f'(c)=f''(c)=f'''(c)=0$, then show that f has a minimum at c.

(9)

(b) Let

$$f(x,y)=\begin{cases}\dfrac{x\left(x^2-y^2\right)}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

(i) Find the first order partial derivatives of f at the point $(0,0)$, if they exist.

(ii) Check for the continuity and the differentiability of f at the point $(0,0)$. (12)

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Q.24 (a) Find the volume of the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $6x + 4y + 3z = 12$. (12)

(b) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 6 & 9 & 18 \\ 1 & 2 & 6 & 12 \end{bmatrix}$$

Find a basis for the null space of A. (9)

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Q.25 Solve the following differential equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=e^{-x}\sin\left(\frac{\pi}{3}-x\right),\quad y(0)=y'(0)=0.$$

(21)

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Space for rough work

Space for rough work

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