IIT-JEE 2010 Mathematics Paper I

PART II - Mathematics

SECTION - I Single Correct Choice Type

This section contains 8 multiple choice questions. Each question has four choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

29. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = e^{x^2} + e^{-x^2}$ and $h(x) = x^2 e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then

(A) $a = b$ and $c \neq b$ (B) $a = c$ and $a \neq b$ (C) $a \neq b$ and $c \neq b$ (D) $a = b = c$

29. **(D)** Since, $f'(x) = 2x(e^{x^2} - e^{-x^2}) > 0 \ \forall \ x \in (0, 1]$

$g'(x) = e^{x^2}(2x^2 - 2x + 1) > 0 \ \forall \ x \in [0, 1]$

$h'(x) = 2x^3 e^{x^2} > 0 \ \forall \ x \in (0, 1]$.

$\therefore$ Hence, all three functions are strictly increasing functions, so $a = b = c = 2e$.

30. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having α / β and β / α as its roots is

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$ (B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$ (D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

30. **(B)** Since $\alpha^3 + \beta^3 = (-p)(p^2 - 3\alpha\beta) = q \Rightarrow \alpha\beta = \dfrac{(p^3 + q)}{3p}$

$\therefore$ Required quadratic equation is $x^2 - \left(\dfrac{p^3 - 2q}{p^3 + q}\right)x + 1 = 0$ as $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{p^2 - 2\alpha\beta}{\alpha\beta}$.

31. Equation of the plane containing the straight line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z}{4}$ and perpendicular to the plane containing the straight lines $\dfrac{x}{3} = \dfrac{y}{4} = \dfrac{z}{2}$ and $\dfrac{x}{4} = \dfrac{y}{2} = \dfrac{z}{3}$ is

(A) $x + 2y - 2z = 0$ (B) $3x + 2y - 2z = 0$ (C) $x - 2y + z = 0$ (D) $5x + 2y - 4z = 0$

31. **(C)** Equation of plane must be $a(x - 0) + b(y - 0) + c(z - 0) = 0$ and $2a + 3b + 4c = 0$.

Also, the plane which contain both lines must have normal along the vector

$(3\hat{i} + 4\hat{j} + 2\hat{k}) \times (4\hat{i} + 2\hat{j} + 3\hat{k}) = 8\hat{i} - \hat{j} - 10\hat{k}$.

Since, required plane is perpendicular to this plane

$\Rightarrow$ $8a - b - 10c = 0$. Using cross multiplication $a : b : c = 1 : -2 : 1$

$\Rightarrow$ $x - 2y + z = 0$.

32. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is

(A) 1 / 2 (B) √3 / 2 (C) 1 (D) √3

32. **(D)** Since, $\angle B = \pi / 3$. Expression given is

$$\frac{2a}{c}\sin C.\cos C + \frac{2c}{a}\sin A.\cos A = \frac{1}{R}(a\cos C + c\cos A) = \frac{b}{R}$$

Using sine rule, $b / R = 2 \sin B = \sqrt{3}$.

33. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair dice is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is

(A) 1 / 18 (B) 1 / 9 (C) 2 / 9 (D) 1 / 36

33. **(C)** Since, we have two choices for all r_1, r_2 and r_3
hence, $(2 \times 2 \times 2 \times 3!) / 6^3 = 2 / 9$.

34. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}, 4\hat{i}, 3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively. The quadrilateral $PQRS$ must be a

(A) parallelogram, which is neither a rhombus nor a rectangle
(B) square
(C) rectangle, but not a square
(D) rhombus, but not a square

34. **(A)** Position vectors of mid point of PR and QS both coincides to $\left(\frac{1}{2}\hat{i} + \hat{j}\right)$, hence $PQRS$ is a parallelogram.

But $\overline{PQ}.\overline{PS} = (6\hat{i} + \hat{j}).(-\hat{i} + 3\hat{j}) \neq 0$ and $\overline{PR}.\overline{SQ} = (5\hat{i} + 4\hat{j}).(7\hat{i} - 2\hat{j}) \neq 0$

Hence, $PQRS$ is neither a rectangle nor a rhombus.

35. The value of $\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t \ln(1+t)}{t^4 + 4} dt$ is

(A) 0 (B) 1 / 12 (C) 1 / 24 (D) 1 / 64

35. **(B)** Since $\frac{0}{0}$ form exist, Using L'Hospital rule and Newton-Leibnitz rule,

required limit = $\lim_{x \to 0} \frac{x \ln(1+x)}{3x^2(x^4 + 4)} = 1/12$

36. The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is

(A) 0 (B) $2^9 - 1$ (C) 168 (D) 2

36. **(A)** We have three planes and we have to make sure that they intersect in exactly two points and that is impossible, because system of equations may have no solution or unique solution or infinitely many solutions.

SECTION - II Multiple Correct Choice Type

This section contains 5 multiple choice questions. Each question has four choices (A), (B), (C) and (D) out of which ONE OR MORE may be correct.

37. Let ABC be a triangle such that $\angle ACB = \pi/6$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)
(A) $-(2+\sqrt{3})$ (B) $1+\sqrt{3}$ (C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

37. **(B)** Since, $b = x^2 - 1 \Rightarrow x > 1 \Rightarrow a > c$

$$\therefore \quad \cos A = \frac{(x^2-1)^2 + (2x+1)^2 - (x^2+x+1)^2}{2(x^2-1)(2x+1)} = -1/2$$

$\Rightarrow A = 2\pi/3 \quad \therefore \angle B = \angle C = \pi/6 \quad \Rightarrow b = c \quad \Rightarrow x^2 - 2x - 2 = 0$
$\therefore \quad x = 1 \pm \sqrt{3}$ of which only $x = 1 + \sqrt{3}$ is acceptable.

38. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be
(A) $-1/r$ (B) $1/r$ (C) $2/r$ (D) $-2/r$

38. **(CD)** Let $A(t_1)$ and $B(t_2)$, then y-coordinate of centre of the circle with AB as diameter is

$(t_1 + t_2) = r$ and slope of line joining $AB = \dfrac{2}{(t_1+t_2)} = 2/r$.

By symmetry, slope of A' and $B' = -2/r$ where $A'B'$ are image of A and B in axis.

39. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$ is (are)

(A) $\frac{22}{7} - \pi$ (B) $2/105$ (C) 0 (D) $\frac{71}{15} - \frac{3\pi}{2}$

39. **(A)** $$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \int_0^1 \frac{x^8 - 4x^7 + 6x^6 - 4x^5 + x^4}{1+x^2}\,dx$$

$$= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \right) dx$$

$$= (1/7) - (4/6) + 1 - (4/3) + 4 - 4\tan^{-1} 1 = \frac{22}{7} - \pi.$$

40. Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)\,z_1 + t z_2$ for some real number t with $0 < t < 1$. If Arg (w) denotes the principal argument of a non zero complex number w, then
(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ (B) Arg $(z - z_1) =$ Arg $(z - z_2)$
(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$ (D) Arg $(z - z_1) =$ Arg $(z_2 - z_1)$

40. **(ACD)** Using section formula, z_1, z_2 and z are collinear.
Since, $t \in (0, 1)$ hence, z lies between z_1 and z_2 such that z divides line joining z_1 and z_2 into ratio $t : (1-t)$ ratio internally.
Hence, $|z - z_1| + |z - z_2| = |z_1 - z_2|$.

Angle subtended at z_1 by z and z_2 is zero $\Rightarrow \dfrac{z - z_1}{z_2 - z_1}$ must be purely real.

Since, points are collinear then argument must be same if taken in a sequence.

41. Let f be a real valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\,dt$. Then which of the following statement(s) is (are) true?
(A) $f''(x)$ exists for all $x \in (0, \infty)$
(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$
(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (\alpha, \infty)$

41. **(BC)** $f'(x) = \dfrac{1}{x} + \sqrt{1+\sin x}$ which is defined and positive for all x.
It is continuous at all x but not differentiable when $\sin x = -1$ $\Rightarrow$ (B) is correct and (A) is wrong.
For $x > \alpha$, $f'(x) < (1/\alpha) + \sqrt{2}$
While $f(x)$ is monotonically increasing $\Rightarrow$ (C) is correct.
As $x \to \infty$, $f(x) \to \infty$: unbounded. $\Rightarrow$ (D) is incorrect.

SECTION - III (Paragraph Type)

This Section contains 2 paragraphs. Based upon the first paragraph 2 multiple Choice questions and based upon the second paragraph 3 multiple choice questions have to be answered. Each of these questions has four choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

Paragraph for Q.NO 42 to Q.NO. 43

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ intersect at the points A and B.

42. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is
(A) $2x - \sqrt{5}\,y - 20 = 0$ (B) $2x - \sqrt{5}\,y + 4 = 0$ (C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

42. **(B)** Equation of tangent to hyperbola, $y = mx + \sqrt{(a^2m^2 - b^2}$; $y = mx + \sqrt{(9m^2 - 4)}$
Equation of circle $x^2 + y^2 - 8x = 0$
$\Rightarrow (x-4)^2 + y^2 = 4^2$ centre (4, 0) ; radius = 4
$\therefore \{m.4 - 0 + \sqrt{(9m^2 - 4)}\} / \{\sqrt{(1 + m^2)}\} = 4 \Rightarrow 8m\sqrt{(9m^2 - 4)} = 20 - 9m^2$
$m = (2/\sqrt{5})$ (Checking with option)
$\therefore$ Equation of common tangent $y = 2/\sqrt{5}x + \sqrt{\{(9.(4/5)) - 4\}}$ $\therefore 2x - \sqrt{5}\,y + 4 = 0$

43. Equation of the circle with AB as its diameter is
(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$
(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

43. **(A)** Circle $(x-4)^2 + y^2 = 16$, hyperbola $y^2 = (x^2 - 9)(4/9)$
solving we get $x = 6, -6/13$
when $x = 6$ $\therefore y^2 = 12$ when $x = (-6/13)$ $y^2 = -$ve not acceptable
$\therefore$ A and B are $(6, +2\sqrt{3})$ & $(6, -2\sqrt{3})$
$\therefore$ Equation. of circle with diameter AB
$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
$\therefore x^2 + y^2 - 12x + 24 = 0$

Paragraph for Q.NO 44 to Q.NO. 46

Let p be an odd prime number and T_p be the following set of 2×2 matrices :

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a,b,c \in (0,1,2,....p-1 \right\}$$

44. The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det (A) divisible by p is
(A) $(p-1)^2$ (B) $2(p-1)$ (C) $(p-1)^2 + 1$ (D) $2p - 1$

44. **(D)** $b = c$ and $a^2 - b^2$ is divisible by p
$\therefore$ $a + b$ is a multiple of p or $a = b$
$a = b$ has p solutions while $a + b = p$ has $(p - 1)$ solution $\therefore$ Total $= 2p - 1$

45. The number of A in T_p such that trace of A is not divisible by p but det (A) is divisible by p is
[**Note :** The trace of a matrix is the sum of its diagonal entries.]
(A) $(p - 1)\ (p^2 - p + 1)$ (B) $p^3 - (p - 1)^2$
(C) $(p - 1)^2$ (D) $(p - 1)\ (p^2 - 2)$

45. **(C)** a is non-zero. Let a^2 mod $p = K$.
For $b > 0$ b mod p, $2b$ mod p, $3b$ mod p take distinct values in {0, 1, 2, ……, $p - 1$}
$\therefore$ There exists only one c for which bc mod $p = K$.
$\therefore$ For every non-zero a & b, there exists an unique c.
$\therefore$ No. of possibilities $= (p - 1)^2$.

46. The number of A in T_p such that det (A) is not divisible by p is
(A) $2p^2$ (B) $p^3 - 5p$ (C) $p^3 - 3p$ (D) $p^3 - p^2$

46. **(D)** For $a = 0$, det (A) is divisible by p iff at least one of b & c is zero i.e.
$2p - 1$ possibilities.
$\therefore$ $(p - 1)^2 + (2p - 1) = p^2$ cases when det (A) is divisible by p.
$\therefore$ det (A) is not divisible by p in $(p^3 - p^2)$ cases.

SECTION - IV (Integer Type)

This Section contains **TEN questions. The** answer to each question is a **single-digit integer,** ranging from 0 to 9. The correct digit below the question number in the ORS is to be bubbled.

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47. Let f be a real-valued differentiable function on $\boldsymbol{R}$ (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point $P\,(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscissa of P, then the value of $f(-3)$ is equal to

47. **9** Tangent at $p(x, y)$: $Y - y = f'(x)\,(X - x)$
$\Rightarrow f'(x)\,.\,X - Y + y - x f'(x) = 0$
According to question $y - x f'(x) = x^3$ $\Rightarrow f'(x) - (y / x) = -x^2$
$\Rightarrow (dy / dx) - (y / x) = -x^2$ $(\because\ dy / dx = f'(x))$
This is L.D.E. in terms of y
$\therefore$ I.F $= e^{-\int \frac{dx}{x}} = e^{-\ln x} = 1 / x$
Solution is : $y.\,(\text{I.F}) = -\int x^2.\,(\text{I.F})\,dx = -\int x^2.\,(1 / x)dx + k$
$(y / x) = (-\,x^2 / 2) + k$
put $x = 1$ then $1 = -\tfrac{1}{2} + k$ $\Rightarrow k = (3 / 2)$
$\therefore (y / x) = -x^2 / 2 + (3 / 2)$
$\Rightarrow F(x) = (-\,x^3 / 2) + (3x / 2)$ $\therefore F(-3) = (27 / 2) - (9 / 2) = (18 / 2) = 9$

48. The number of value of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that for $\theta \neq \frac{n\pi}{5}$

$n = 0, \pm1, \pm2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is

48. 3 $\because$ $\tan\theta = \cot 5\theta$

$\Rightarrow$ $\tan 5\theta \tan\theta = 1$

$\Rightarrow$ $\cos(5\theta + \theta) = 0$

$\Rightarrow$ $6\theta = (2n+1)(\pi/2)$

$\Rightarrow$ $\theta = (2n+1)(\pi/12),\ n \in I$

For $n = 0, 1, 2, -1, -2$ then $\theta = (\pi/12), (\pi/4), (5\pi/12), (-\pi/12), (-\pi/4)$

Again, $\sin 2\theta = \cos 4\theta$

$\Rightarrow \sin 2\theta - \sin\{(\pi/2) - 4\theta\} = 0$

$\Rightarrow 2\cos\{2\theta + (\pi/2) - 4\theta)/2 \,.\, \sin(6\theta - (\pi/2)/2) = 0$

$\therefore$ $\cos\{(\pi/4) - \theta\} = 0$

$\Rightarrow$ $\{(\pi/4) - \theta\} = (2n+1)(\pi/2),\ n \in I$

$\Rightarrow$ $\theta = \pi/4 - (2n+1)(\pi/2)$

For $n = 0, 1, 2, -1, -2$ then $\theta = -(\pi/4), -(5\pi/4), -(9\pi/4), (3\pi/4), (7\pi/4)$

and $\sin(3\theta - (\pi/4) = 0$

$\Rightarrow$ $3\theta - (\pi/4) = n\pi$ $\quad\Rightarrow \theta = \{(n\pi + (\pi/4)/3\},\ n \in I$

For $n = 0, 1, 2, -1, -2$ then $\theta = (\pi/12), (5\pi/12), (39\pi/12), (-3\pi/12), (-7\pi/12)$

Hence, common solution is :

$\theta = (-\pi/4), (5\pi/12), (\pi/12)$ $\qquad\therefore n = 3.$

49. The maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is

49. 2. Since, $\frac{1}{1 + 4\cos^2\theta + \frac{3}{2}\sin 2\theta} = \frac{1}{1 + 2(1+\cos^2\theta) + \frac{3}{2}\sin 2\theta} = \frac{1}{3 + \frac{3}{2}\sin 2\theta + 2\cos 2\theta}$

Maximum value when denominator is minimum.

Minimum value of $\frac{3}{2}\sin 2\theta + 2\cos 2\theta = -\sqrt{\frac{9}{4} + 4} = -5/2$

$\therefore$ Maximum value = $1/(3 - (5/2)) = 2$.

50. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a} = \frac{\vec{i} - 2\vec{j}}{\sqrt{5}}$ and $\vec{b} = \frac{2\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{14}}$, then the value

of $(2\vec{a} + \vec{b}).\left[(\vec{a} \times \vec{b}) \times (\vec{a} - 2\vec{b})\right]$ is

50. **5.** Given $\vec{a} = \frac{\hat{i}}{\sqrt{5}} - \frac{2\hat{j}}{\sqrt{5}}, \vec{b} = \frac{3\hat{i}}{\sqrt{14}} + \frac{\hat{j}}{\sqrt{14}} + \frac{3\hat{k}}{\sqrt{14}}$

Now, $(2\vec{a}+\vec{b}).\{(\vec{a}\times\vec{b})\times(\vec{a}-2\vec{b})\} = [(2\vec{a}+\vec{b})\ (\vec{a}\times\vec{b})(\vec{a}-2\vec{b})]$

$= -(\vec{a}\times\vec{b}).\{(2\vec{a}+\vec{b})\times(\vec{a}-2\vec{b})\}$

$= -(\vec{a}\times\vec{b}).\{2\vec{a}\times\vec{a} - 4\vec{a}\times\vec{b} + \vec{b}\times\vec{a} - 2\vec{b}\times\vec{b}\}$

$$= 5(\vec{a}\times\vec{b}).(\vec{a}\times\vec{b}) = 5|\vec{a}\times\vec{b}|^2 = 5\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 2/\sqrt{14} & 1/\sqrt{14} & 3/\sqrt{14} \end{vmatrix}^2$$

$$= 5\left|\left\{-\frac{6}{\sqrt{70}}\hat{i} + \hat{j}\left(-\frac{3}{\sqrt{70}}\right) + \hat{k}\left(\frac{1}{\sqrt{70}} + \frac{4}{\sqrt{70}}\right)\right\}\right|^2$$

$= (5/70)(36 + 9 + 25) = 5.$

51. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is

51. **2.** Intersection point of directrics with x-axis is $(a/e, 0)$

$\because$ $2x + y = 1$ passes through $(a/e, 0)$ $\Rightarrow$ $a/e = 1/2$.

$\therefore$ $m = -2$

$\Rightarrow$ $c = \sqrt{(a^2m^2 - b^2)}$ $\Rightarrow$ $1 = \sqrt{(4a^2 - b^2)}$

$\Rightarrow$ $4a^2 - b^2 = 1$ $\Rightarrow$ $4a^2 - a^2(e^2 - 1) = 1$

$\Rightarrow$ $4 - e^2 + 1 = 1$ $\Rightarrow$ $e^2 = 4$ $\Rightarrow$ $e = 2$.

52. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \text{ is } \sqrt{6}$$, then $|d|$ is

52. **6.** $\vec{n} = \vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\hat{i} + 2\hat{j} - \hat{k}$

Equation of plane containing the lines is $\vec{r}\ .\ \vec{n} = \vec{a}\ .\ \vec{n}$

i.e. $-x - 2y - z = 0$

$\therefore$ distance between $Ax - 2y + z - d = 0$ and $-x - 2y - z = 0$

$= |d| / \sqrt{(1 + 4 + 1)} = \sqrt{6}$

$\Rightarrow$ $|d| = 6$.

53. For any real number x, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [– 10 , 10] by

$$f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd}, \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx$ is

53. **4.** Since $f(x)$ is an even periodic function with period 2. Then given integral = $\pi^2\int_0^2 f(x)\cos \pi x\, dx$

$$= \pi^2\left[\int_0^1 (1-x)\cos \pi x\, dx + \int_1^2 (x-1)\cos \pi x\, dx\right]$$

$$= \pi^2\left[\int_0^1 (1-x)\cos \pi x\, dx - \int_0^1 x\cos \pi x\, dx\right]$$

$$= \pi^2\left[\int_0^1 (1-2x)\cos \pi x\, dx\right] = \pi^2 \times (4/\pi^2) = 4.$$

54. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to

54. **1.** Since $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ Using $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow (z+1+\omega+\omega^2)\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$$\therefore z = 0 \text{ and } \begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & z+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & z+\omega-\omega^2 \end{vmatrix} = 0$$

$\Rightarrow \{z-(\omega-\omega^2)\ (z+\omega-\omega^2)\} - (1-\omega^2)(1-\omega) = 0$

$\Rightarrow z^2 - (\omega-\omega^2)^2 - (1+\omega^3-\omega-\omega^2) = 0$

$\Rightarrow z^2 - (\omega^2+\omega^4-2\omega\omega^2) - (2+1) = 0$

$\Rightarrow z^2 = 0 \quad \Rightarrow \quad z = 0.$

55. Let S_K, k = 1, 2 ...... 100, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100}\left|\left(k^2-3k+1\right)S_k\right|$ is

55. **4.** Given, first term, $a = \frac{k-1}{k!}$, $r = 1/k$

$$\therefore S_k = \frac{(k-1)/k!}{1-(1/k)} = k/k! = \frac{1}{(k-1)!}$$

$$\text{Now, } \frac{100^2}{100!} + \sum_{k=1}^{100} |(k^2 - 3k + 1)S_k| = \frac{100^2}{100!} + \sum_{k=1}^{100} \left|\frac{k^2 - 3k + 1}{(k-1)!}\right|$$

$$= \frac{100^2}{100!} + \sum_{k=1}^{100} \left|\frac{(k^2 - 2k + 1) - k}{(k-1)!}\right| = \frac{100^2}{100!} + \sum_{k=1}^{100} \left|\frac{(k-1)^2}{(k-1)!} - \frac{k}{(k-1)!}\right| = 4.$$

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56. The number of all possible value of θ, where $0 < \theta < \pi$, for which the system of equations

$(y + z) \cos 3\theta = (xyz) \sin 3\theta$

$x \sin 3\theta = \dfrac{2\cos 3\theta}{y} + \dfrac{2\sin 3\theta}{z}$

$(xyz) \sin 3\theta = (y + 2z) \cos 3\theta + y \sin 3\theta$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is

56. **8.** Since, $xyz \sin 3\theta - y \cos 3\theta - z \cos 3\theta = 0$

$xyz \sin 3\theta - 2y \sin 3\theta - 2z \cos 3\theta = 0$

and $xyz \sin 3\theta - y (\sin 3\theta + \cos 3\theta) - 2z \cos 3\theta = 0$

Eliminating xyz , y and z , we get

$$\Rightarrow \begin{vmatrix} \sin 3\theta & -\cos 3\theta & -\cos 3\theta \\ \sin 3\theta & -2\sin 3\theta & -2\cos 3\theta \\ \sin 3\theta & -(\sin 3\theta + \cos 3\theta) & -2\cos 3\theta \end{vmatrix} = 0$$

$$\Rightarrow \quad \sin 3\theta \, . \, \cos 3\theta \begin{vmatrix} 1 & \cos 3\theta & 1 \\ 1 & 2\sin 3\theta & 2 \\ 1 & \sin 3\theta + \cos 3\theta & 2 \end{vmatrix} = 0$$

$\therefore$ $\sin 3\theta = 0 \Rightarrow \theta = n\pi / 3, n \in I \Rightarrow \theta = \pi / 3, 2\pi / 3$

$\cos 3\theta = 0 \Rightarrow \theta = (2n + 1)(\pi / 6), n \in I \Rightarrow \theta = \pi / 6, \pi / 2, 5\pi / 6$

and $(4 \sin 3\theta - 2 \sin 3\theta - 2 \cos 3\theta) + \cos 3\theta (2 - 2) + (\sin 3\theta + \cos 3\theta - 2 \sin 3\theta) = 0$

$\Rightarrow$ $(2 \sin 3\theta - 2 \cos 3\theta) + (\cos 3\theta - \sin 3\theta) = 0$

$\Rightarrow$ $\sin 3\theta - \cos 3\theta = 0$

$\Rightarrow$ $\tan 3\theta = 1 \Rightarrow 3\theta = n\pi + (\pi / 4)$

$\Rightarrow$ $\theta = (n\pi / 3) + (\pi / 12) = 3\pi / 4, n \in I.$

$\Rightarrow$ $\theta = \pi / 12, 5\pi / 12, 3\pi / 4.$

$\therefore$ Number of solutions = 8.