HIGHER SECONDARY SECOND YEAR PUBLIC EXAMINATION

MARCH– 2024

MATHEMATICS – ANSWER KEY

PART-I

Note: i) Answer all the questions. $[20 \times 1 = 20]$

ii) Choose the most appropriate answer from the given four alternatives and write the option code and the corresponding answer

	TYPE-A		TYPE-B	
1.	**(a)**	$\frac{8}{3}$	**(a)**	$45°$
2.	**(d)**	$\frac{3\pi a^4}{16}$	**(d)**	8
3.	**(a)**	10	**(a)**	$\frac{8}{3}$
4.	**(d)**	2	**(a)**	10
5.	**(d)**	8	**(d)**	$\frac{1}{\sqrt{5}}$
6.	**(a)**	$[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$	**(c)**	-4
7.	**(d)**	$\frac{d^2y}{dx^2} - y = 0$	**(b)**	$-\frac{q}{r}$
8.	**(b)**	$y = 0$	**(d)**	$\frac{3\pi a^4}{16}$
9.	**(d)**	$\frac{1}{(x+1)^2}dx$	**(d)**	$adj(AB) = (adj\ A)(adj\ B)$
10.	**(a)**	$x^2 + y^2$	**(a)**	$[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$
11.	**(c)**	2	**(d)**	$\frac{d^2y}{dx^2} - y = 0$
12.	**(d)**	Parabola	**(c)**	2
13.	**(c)**	$\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$	**(d)**	Parabola
14.	**(a)**	$45°$	**(d)**	2
15.	**(d)**	$\frac{1}{\sqrt{5}}$	**(c)**	$\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$
16.	**(b)**	$-\frac{q}{r}$	**(a)**	$x^2 + y^2$
17.	**(a)**	0	**(b)**	$y = 0$
18.	**(c)**	2	**(a)**	0
19.	**(d)**	$adj(AB) = (adj\ A)(adj\ B)$	**(c)**	2
20.	**(a)**	-4	**(d)**	$\frac{1}{(x+1)^2}dx$

PART-II

Note: $[7 \times 2 = 14]$

(i) Answer any **SEVEN** questions

(ii) Question number **30** is compulsory.

21. Simplify: $\sum_{n=1}^{12} i^n$.

Solution:

$$\sum_{n=1}^{12} i^n = (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + (i^9 + i^{10} + i^{11} + i^{12})$$
$$= (i - 1 - i + 1) + (i - 1 - i + 1) + (i - 1 - i + 1)$$
$$= 0 + 0 + 0 = 0$$

22. If α and β are the roots of the quadratic equation $2x^2-7x+13=0$, construct a quadratic equation whose roots are α^2 and β^2.

Solution:

$$\text{G.T. } \alpha \text{ and } \beta \text{ are the roots of } 2x^2-7x+13=0, \text{ then}$$

$$\alpha+\beta=\frac{7}{2} \text{ and } \alpha\beta=\frac{13}{2}$$

$$S.R.=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta = \left(\frac{7}{2}\right)^2-2\left(\frac{13}{2}\right)$$

$$=\frac{49}{4}-13=\frac{49-52}{4}=-\frac{3}{4}$$

$$P.R.=\alpha^2\beta^2=(\alpha\beta)^2=\left(\frac{13}{2}\right)^2=\frac{169}{4}$$

The quadratic equation is,

$$x^2-(S.R.)x+P.R.=0$$

$$x^2-\left(-\frac{3}{4}\right)x+\frac{169}{4}=0$$

$$\boxed{4x^2+3x+169=0} \text{ is the req. quadratic equation}$$

23. Find df for $f(x)=x^2+3x$ and evaluate it for $x=3$ and $dx=0.02$

Solution:

$$\text{G.T } f(x)=x^2+3x$$

$$f'(x)=2x+3$$

$$\text{W.k.T. } df=f'(x)dx$$

$$df=(2x+3)dx$$

$$x=3 \text{ and } dx=0.02$$

$$df=(2(3)+3)(0.02)$$

$$df=(9)(0.02)$$

$$df=0.18$$

24. Find the differential equation for the family of all straight lines passing through the origin.

Solution:

The family of straight lines passing through the origin is $y=mx \to ①$

Equ ① diff. w.r.t. to 'x' $\frac{dy}{dx}=m \to ②$

$$\frac{dy}{dx}=m \to ②$$

Sub ② in ①, we get

$$y=\frac{dy}{dx}x \Rightarrow x\frac{dy}{dx}-y=0 \text{ is the required DE.}$$

$$\Rightarrow x\frac{dy}{dx}-y=0 \text{ is the required DE.}$$

25. For the random variable X with the given probability mass function as below, $f(x)=\begin{cases}2(x-1), & 1<x<2\\ 0 & \text{otherwise}\end{cases}$ find the mean.

Solution:

$$\text{mean}=E(x)=\int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$=\int_1^2 x\big(2(x-1)\big)\,dx=2\int_1^2 (x^2-x)\,dx=2\left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^2$$

$$=2\left[\left(\frac{8}{3}-\frac{4}{2}\right)-\left(\frac{1}{3}-\frac{1}{2}\right)\right]=2\left[\frac{7}{3}-\frac{3}{2}\right]=2\left[\frac{14-9}{6}\right]$$

$$=\frac{5}{3}$$

26. Find the general equation of a circle with centre $(-3,-4)$ and radius 3 units.

Solution:

$$\text{Here, } C(h,k) = C(-3,-4) \text{ and } r = 3$$

Equaion of circle is

$$(x-h)^2 + (y-k)^2 = r^2$$
$$(x+3)^2 + (y+4)^2 = 3^2$$
$$x^2 + 6x + 9 + y^2 + 8y + 16 = 9$$
$$x^2 + y^2 + 6x + 8y + 16 = 0$$

27. Find the rank of the matrix $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$ by minor method.

Solution:

$$\text{Let } A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$$
$$\rho(A) \le \min\{3,2\}$$
$$\rho(A) \le 2$$

Consider, 2^{nd} order minor of A

$$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = -5 \ne 0$$
$$\therefore \boxed{\rho(A) = 2}$$

28. Evaluate $\int_0^{\frac{\pi}{2}} \sin^{10} x \; dx$.

Solution:

$$I_{10} = \int_0^{\frac{\pi}{2}} \sin^{10} x \; dx = \frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{63\pi}{512} \qquad n \text{ is even}$$

29. Evaluate : $\lim\limits_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right)$.

Solution:

$$\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right) \qquad \left(\frac{0}{0}\text{form}\right)$$

Appling l'Hôpital Rule, we get

$$= \lim_{x\to 1}\left(\frac{2x-3}{2x-4}\right)$$
$$= \frac{2(1)-3}{2(1)-4} = \frac{-1}{-2} = \frac{1}{2}$$
$$\therefore \lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right) = \frac{1}{2}$$

30. Show that the vectors $2\vec{i} - \vec{j} + 3\vec{k}, \vec{i} - \vec{j}$ and $3\vec{i} - \vec{j} + 6\vec{k}$ are coplanar.

Solution:

$$\text{Let } \vec{a} = 2\vec{i} - \vec{j} + 3\vec{k},\ \vec{b} = \vec{i} - \vec{j},\ \vec{c} = 3\vec{i} - \vec{j} + 6\vec{k}$$
$$\text{W.K.T. If } \vec{a}, \vec{b}, \vec{c} \text{ are coplanar} \Leftrightarrow [\vec{a}, \vec{b}, \vec{c}] = 0$$
$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = 1(1-2) - 2(-2-6) - 3(2+3)$$
$$= -1 + 16 - 15 = 0$$

∴ The given three vectors are coplanar

PART-III

Note: [**7 × 3 = 21**]

(i) Answer any **SEVEN** questions

(ii) Question number **40** is compulsory.

31. Show that $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1} x, \ |x| > 1$.

Solution:

$$\text{Let } \theta = \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) \rightarrow (1)$$

$$\cot\theta = \frac{1}{\sqrt{x^2-1}} = \frac{\text{adj}}{\text{opp}}$$

$$\text{adj} = 1, \quad \text{opp} = \sqrt{x^2-1},$$

$$\text{hyp} = \sqrt{\text{opp}^2 + \text{adj}^2} = \sqrt{1^2 + \left(\sqrt{x^2-1}\right)^2} = \sqrt{1 + x^2 - 1} = x$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{x}{1} = x$$

$$\theta = \sec^{-1} x \rightarrow (2)$$

From (1) & (2), we get

$$\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1} x$$

32. Find the equations of tangent and normal to the parabola $x^2 + 6x + 4y + 5 = 0$ at $(1, -3)$.

Solu ion

$$x^2 + 6x + 4y + 5 = 0$$

Equ. of tangent at (x_1, y_1) is

$$xx_1 + 6\frac{1}{2}(x + x_1) + 4\frac{1}{2}(y + y_1) + 5 = 0$$

$$xx_1 + 3(x + x_1) + 2(y + y_1) + 5 = 0$$

Equ. of tangent at $(1, -3)$ is

$$x(1) + 3(x + 1) + 2(y - 3) + 5 = 0$$

$$x + 3x + 3 + 2y - 6 + 5 = 0$$

$$4x + 2y + 2 = 0$$

$\div$ by 2

$$2x + y + 1 = 0 \text{ is a equ. of tangent}$$

Equ. of normal is of the form $x - 2y + k = 0 \rightarrow (1)$

At $(1, -3)$

$$1 - 2(-3) + k = 0 \Longrightarrow k = -7$$

$\therefore$ Equ. of normal is $x - 2y - 7 = 0$

33. Prove that $\left[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}\right] = 0$

Solution:

$$\left[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}\right] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} \left[\vec{a}, \vec{b}, \vec{c}\right]$$

$$= \{1(1-0) + 1(0-1) + 0\}\left[\vec{a}, \vec{b}, \vec{c}\right]$$

$$= \{1 - 1 + 0\}\left[\vec{a}, \vec{b}, \vec{c}\right]$$

$$\therefore \left[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}\right] = 0$$

34. If $u(x, y) = \frac{x^2+y^2}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$.

Solution:

$$u(x, y) = \frac{x^2 + y^2}{\sqrt{x+y}}$$

$$u(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{\sqrt{(\lambda x) + (\lambda y)}} = \frac{\lambda^2(x^2 + y^2)}{\sqrt{\lambda(x+y)}} = \frac{\lambda^2}{\lambda^{\frac{1}{2}}}\frac{x^2 + y^2}{\sqrt{x+y}} = \lambda^{\frac{3}{2}} u(x, y)$$

$\therefore u(x, y)$ is homogeneous function with degree $\frac{3}{2}$

By Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$$

35. Find two positive numbers whose sum is 12 and their product is maximum.

Solution:

Let x, y be the two positive numbers

$$\text{G.T.}\quad x + y = 12 \implies y = 12 - x \to (1)$$

$$\text{Product} = xy$$

$$P(x) = x(12 - x) = 12x - x^2$$

$$P'(x) = 12 - 2x$$

$$P''(x) = -2$$

$$P'(x) = 0$$

$$12 - 2x = 0$$

$$-2x = -12$$

$$x = 6$$

Sub $x = 6$ in $P''(x)$

$$P''(6) = -2 < 0$$

$\therefore$ $P(x)$ is local maximum at $x = 6$

i.e., Product is maximum at $x = 6$

Sub $x = 6$ in (1)

$$y = 12 - 6 = 6$$

Hence, the two positive numbers are 6, 6

36. Evaluate $\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}} dx$ using properties of integration

Solution:

$$\text{Let } I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}} dx \to (1)$$

$$I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan\left(\frac{3\pi}{8}+\frac{\pi}{8}-x\right)}} dx \quad \because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}} dx = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\cot x}} dx = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\frac{1}{\sqrt{\tan x}}} dx$$

$$\therefore I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} dx \to (2)$$

$$(1) + (2) \implies 2I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1+\sqrt{\tan x}}{1+\sqrt{\tan x}} dx = [x]_{\frac{\pi}{8}}^{\frac{3\pi}{8}} = \frac{3\pi}{8} - \frac{\pi}{8}$$

$$2I = \frac{2\pi}{8} \implies I = \frac{\pi}{8}$$

$$\boxed{\therefore \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}} dx = \frac{\pi}{8}}$$

37. Simplify $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$ into rectangular form

Solution:

$$\text{Consider } \frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1-1)+i(1+1)}{1^2+(-1)^2} = \frac{0+2i}{1+1} = \frac{2i}{2} = i$$

and $\frac{1-i}{1+i} = \left(\frac{1+i}{1 \quad i}\right)^{-1} = (i)^{-1} = -i$

$\therefore \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = (i)^3 - (-i)^3 = -i - (+i) = -i - i = -2i = 0 - 2i$

38. Solve $(1+x^2)\frac{dy}{dx} = 1 + y^2$.

Solution:

$$(1+x^2)dy = 1 + y^2$$
$$\frac{1}{1+y^2}dy = \frac{1}{1+x^2}dx$$
$$\int \frac{1}{1+y^2}dy = \int \frac{1}{1+x^2}dx$$
$$\tan^{-1} y = \tan^{-1} x + c$$
$$\tan^{-1} y - \tan^{-1} x = c$$
$$\tan^{-1}\left(\frac{y-x}{1+xy}\right) = c$$
$$\frac{y-x}{1+xy} = \tan c = a \text{ (say)}$$
$$y - x = a(1+xy)$$

39. Three fair coins are tossed simultaneously. Find the probability mass function for number of heads occurred

Solution:

$$S = \{H,T\} \times \{H,T\} \times \{H,T\} = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let $X : S \longrightarrow \mathbb{R}$ be the number of heads, then

The X be a random variables that takes the values 0, 1, 2 & 3Number of elements in inverse images

Number of elements in inverse images

Values of the Random Variable	0	1	2	3	Total
Number of elements in inverse image	1	3	3	1	8

The probabilities are given by

$$f(0) = P(X=0) = \frac{1}{8} \; ; \; f(1) = P(X=1) = \frac{3}{8}$$
$$f(2) = P(X=2) = \frac{3}{8} \; ; \; f(3) = P(X=3) = \frac{1}{8}$$

The function f(x) satisfies the conditions (i) $f(x) \geq 0$, for $x = 0,1,2,3$

$$\text{(ii)} \sum_x f(x) = \sum_{x=0}^{2} f(x) = f(0) + f(1) + f(2) + f(3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

The probability mass function is given by

X	0	1	2	3
f(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

40. If $A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$, then find $|adj(adj\ A)|$.

Solution:

$$\text{w.k.t.} \quad \boxed{|adj(adj\ A)| = |A|^{(n-1)^2}}$$

$$\text{Here } n = 3, \; |adj(adj\ A)| = |A|^{(3-1)^2} \Longrightarrow |adj(adj\ A)| = |A|^4$$

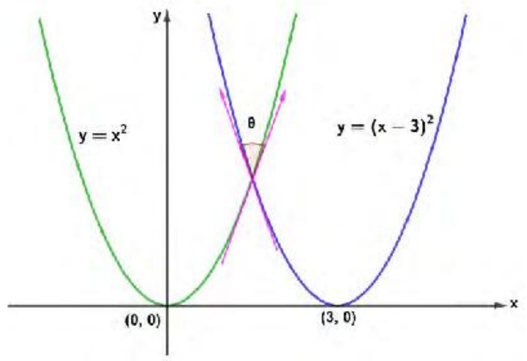
$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix} = 2(9-2) + 1(-15+3) + 3(-10+9)$$
$$= 2(7) + 1(-12) + 3(-1) = 14 - 12 - 3 = 1$$
$$|adj(adj\ A)| = 1$$

PART-IV

ANSWER ALL QUESTIONS. [7 × 5 = 35]

41. (a) Find the acute angle between $y = x^2$ and $y = (x-3)^2$.

Solution:



$$y = x^2 \rightarrow (1)$$
$$y = (x-3)^2 \rightarrow (2)$$

From (1) and (2)

$$x^2 = (x-3)^2$$
$$x^2 = x^2 - 6x + 9$$
$$6x = 9 \Rightarrow x = \frac{3}{2}$$

Sub $x = \frac{3}{2}$ in (1) $\Rightarrow y = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

$\therefore$ The point of intersection is $\left(\frac{3}{2}, \frac{9}{4}\right)$

$$y = x^2$$
$$\frac{dy}{dx} = 2x$$
$$m_1 = \left(\frac{dy}{dx}\right)_{\left(\frac{3}{2},\frac{9}{4}\right)} = 2\left(\frac{3}{2}\right) = 3$$
$$\boxed{m_1 = 3}$$

$$y = (x-3)^2$$
$$\frac{dy}{dx} = 2(x-3)$$
$$m_2 = \left(\frac{dy}{dx}\right)_{\left(\frac{3}{2},\frac{9}{4}\right)} = 2\left(\frac{3}{2} - 3\right) = 2\left(-\frac{3}{2}\right) = -3$$
$$\boxed{m_2 = -3}$$

Let θ be the angle between two curves, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{3-(-3)}{1+(3)(-3)}\right| = \left|\frac{3+3}{1-9}\right| = \left|\frac{6}{-8}\right| = \left|\frac{3}{-4}\right| = \frac{3}{4}$$

$$\Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{3}{4}\right)}$$

(OR)

(b) Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$.

Solution:

$$\text{w.k.t } \tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$
$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$
$$\tan^{-1}\left(\frac{\left(\frac{x-1}{x-2}\right) + \left(\frac{x+1}{x+2}\right)}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right) = \frac{\pi}{4}$$
$$\frac{\left(\frac{x-1}{x-2}\right) + \left(\frac{x+1}{x+2}\right)}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)} = \tan\frac{\pi}{4}$$
$$\frac{\frac{(x-1)(x+2) + (x+1)(x-2)}{(x-2)(x+2)}}{\frac{(x-2)(x+2) - (x-1)(x+1)}{(x-2)(x+2)}} = 1$$
$$\frac{2x^2 - 4}{-3} = 1$$
$$2x^2 - 4 = -3 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

42. **(a)** A six-sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find (i) the probability mass function (ii) the cumulative distribution function (iii) $P(4 \le X < 10)$

Solution:

The sample space

$$S = \{1, 3, 3, 5, 5, 5\} \times \{1, 3, 3\ \ 5, 5, 5\}$$

$$S = \begin{Bmatrix} (1,1), (1,3), (1,3), (1,5), (1,5), (1,5) \\ (3,1), (3,3), (3,3), (3,5), (3,5), (3,5) \\ (3,1), (3,3), (3,3), (3,5), (3,5), (3,5) \\ (5,1), (5,3), (5,3), (5,5), (5\ 5), (5,5) \\ (5,1), (5,3), (5,3), (5,5), (5,5), (5,5) \\ (5,1), (5,3), (5,3)\ \ (5,5), (5,5), (5,5) \end{Bmatrix}$$

Let $X : S \longrightarrow \mathbb{R}$ be the number of total score of two dice

$$\text{i.e.}, X(\alpha, \beta) = \alpha + \beta$$

$$X(1,1) = 1 + 1 = 2$$

$$X(1,3) = X(3,1) = 4$$

$$X(3,3) = 6$$

$$X(3,5) = X(5,3) = 8$$

$$X(5,5) = 10$$

The random variable X takes on the values 2, 4, 6, 8, 10.

Number of elements in inverse images

Values of the Random Variable	2	4	6	8	10	Total
Number of elements in inverse image	1	4	10	12	9	36

The probabilities are given by

$$f(2) = P(X = 2) = \frac{1}{36}\ ;\quad f(4) = P(X = 4) = \frac{4}{36}$$

$$f(6) = P(X = 6) = \frac{10}{36}\ ;\quad f(8) = P(X = 8) = \frac{12}{36}$$

$$f(10) = P(X = 10) = \frac{9}{36}$$

(i) The probability mass function is given by

X	2	4	6	8	10
f(x)	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

(ii) Cumulative distribution function

$$\text{W.K.T.}\quad F(x) = P(X \le x)$$

The cumulative distribution function is

X	2	4	6	8	10
F(x)	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	1

(iii) $P(4 \le X < 10) = P(X = 4) + P(X = 6) + P(X = 8)$

$$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36} = \frac{13}{18}$$

(OR)

(b) If $z = x + iy$ is a complex number such that $\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0$, show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$.

Solution:

$$\text{G.T.}\quad z = x + iy$$

$$\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0$$

$$\text{Im}\left(\frac{2(x+iy)+1}{i(x+iy)+1}\right) = 0$$

$$\text{Im}\left(\frac{2x+i2y+1}{ix-y+1}\right) = 0$$

$$\text{Im}\left(\frac{(2x+1)+i2y}{(1-y)+ix}\right) = 0$$

$$\text{Im}\left(\frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}\right) = 0$$

$$\Rightarrow \frac{2y(1-y) - x(2x+1)}{(1-y)^2 + x^2} = 0$$

$$\Rightarrow 2y(1-y) - x(2x+1) = 0$$

$$\Rightarrow 2y - 2y^2 - 2x^2 - x = 0$$

$\times^{ly}$ by $-$

$$\Rightarrow 2x^2 + 2y^2 + x - 2y = 0, \quad \text{Hence proved.}$$

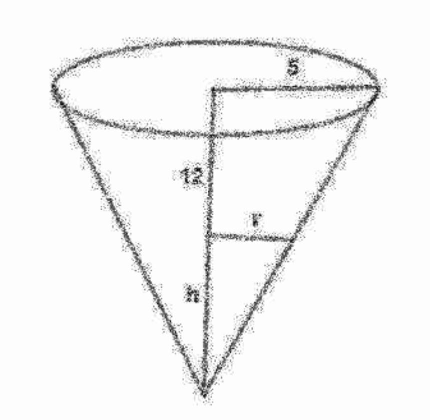
43. (a) A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

Solution:

Let r and h are the radius and height of the conical water tank respectively.

$$\text{Given } r = 5, \quad h = 12 \quad \text{and} \quad \frac{dV}{dt} = 10\ m^3/\text{min}$$

$$\text{Find } \frac{dh}{dt} = ? \quad \text{when } h = 8$$



From diagram,

$$\Rightarrow \frac{r}{5} = \frac{h}{12} \quad \Rightarrow \quad r = \frac{5h}{12}$$

$$\text{w.k.t}\quad V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \frac{25h^2}{144} h$$

$$\therefore V = \frac{25\,\pi\,h^3}{3 \times 144}$$

$$\frac{dV}{dt} = \frac{25\pi}{3 \times 144}(3h^2)\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{25\pi}{144}(h^2)\frac{dh}{dt}$$

$$10 = \frac{25\pi}{144}(64)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{10 \times 144}{25\pi \times 64} = \frac{9}{10\pi}$$

$$\boxed{\frac{dh}{dt} = \frac{9}{10\pi}\,m/\text{min}}$$

(OR)

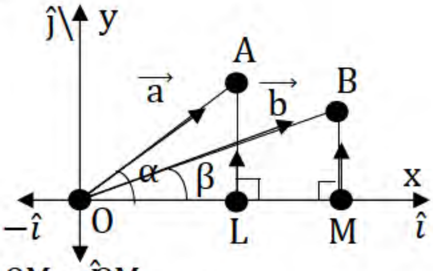
(b) Prove by vector method that $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$.

Solution:

Let $\hat{a}=\overrightarrow{OA}$ and $\hat{b}=\overrightarrow{OB}$ be the unit vectors and which makes angle α and β with positive x − axis respectively.

$\therefore$ The angle between $\hat{a}$ and $\hat{b}$ is $\alpha-\beta$

From the diagram



$$\cos\alpha=\frac{OL}{OA}=\frac{OL}{1}=OL \qquad \cos\beta=\frac{OM}{OB}=\frac{OM}{1}=OM$$

$$\sin\alpha=\frac{LA}{OA}=\frac{LA}{1}=LA \qquad \sin\beta=\frac{MB}{OB}=\frac{MB}{1}=MB$$

$$\hat{a}=\overrightarrow{OA}=\overrightarrow{OL}+\overrightarrow{LA}=OL\,\hat{\imath}+LA\,\hat{\jmath}$$
$$\hat{a}=\cos\alpha\,\hat{\imath}+\sin\alpha\,\hat{\jmath}$$

$$\hat{b}=\overrightarrow{OB}=\overrightarrow{OM}+\overrightarrow{MB}=OM\,\hat{\imath}+MB\,\hat{\jmath}$$
$$\hat{b}=\cos\beta\,\hat{\imath}+\sin\beta\,\hat{\jmath}$$

$$\hat{b}\times\hat{a}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \cos\beta & \sin\beta & 0\\ \cos\alpha & \sin\alpha & 0\end{vmatrix}$$

$$\hat{b}\times\hat{a}=\hat{k}(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\quad\rightarrow ①$$

$$\hat{b}\times\hat{a}=|\hat{b}||\hat{a}|\sin(\alpha-\beta)\,\hat{k}=(1)(1)\sin(\alpha-\beta)\,\hat{k}=\hat{k}\sin(\alpha-\beta)\rightarrow ②$$

From ① and ②, we get

$$\hat{k}\sin(\alpha-\beta)=\hat{k}(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

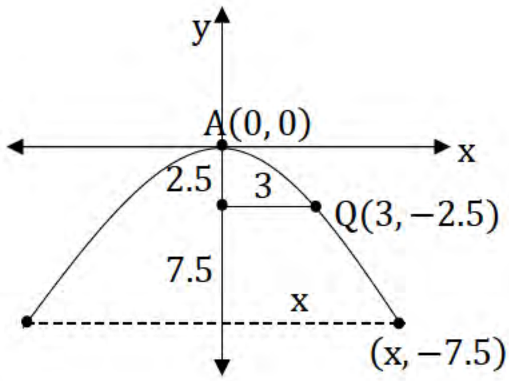
44. (a) Assume that water issuing from the end of a horizontal pipe, 7.5m above the ground, describes parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

Solution:

Let the vertex is A(0, 0) and the parabola is open down then the equ. of parabola is $x^2=-4ay\quad\rightarrow ①$

It passes through (3, 2.5)

$$3^2=-4a(-2.5)$$
$$9=10a$$
$$a=\frac{9}{10}$$



Sub $a=\frac{9}{10}$ in ① we get

$$x^2=-4\left(\frac{9}{10}\right)y\quad\rightarrow ②$$

It passes through (x, −7.5)

$$x^2=-4\left(\frac{9}{10}\right)(-7.5)$$
$$x^2=9\times3$$
$$x=\sqrt{9\times3}$$
$$x=3\sqrt{3}$$

$\therefore$ The water strike the ground beyond vertical line $3\sqrt{3}$ m

(OR)

(b) Solve the Linear differential equation $\frac{dy}{dx} + \frac{y}{x} = \sin x$.

Solution:

$$\frac{dy}{dx} + \frac{y}{x} = \sin x$$

$$\text{Here } P = \frac{1}{x}, Q = \sin x$$

$$\int P\,dx = \int \frac{1}{x}dx = \log|x|$$

$$I.F = e^{\int Pdx} = e^{\log|x|} = x$$

$$\text{The solution is } y(I.F) = \int Q(I.F)dx + c$$

$$y(x) = \int x\sin x\,dx + c$$

$$xy = x(-\cos x) - (1)(-\sin x) + c$$

$$xy = -x\cos x + \sin x + c$$

$$xy + x\cos x = \sin x + c$$

45. (a) The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

Solution:

Let x be the number of bacteria at any time t.

$$\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = kx$$

$$\frac{dx}{x} = kdt$$

$$\int \frac{dx}{x} = k\int dt$$

$$\log|x| = kt + \log|C|$$

$$x = C\,e^{kt} \rightarrow ①$$

$$\text{When } t = 0,\ x = x_0 (\text{initial})$$

$$x_0 = C\,e^0 = C(1)$$

$$\boxed{C = x_0}$$

$$\text{Sub } C = x_0 \text{ in } ①$$

$$x = x_0\,e^{kt} \rightarrow ②$$

$$\text{When } t = 5,\ x = 3x_0$$

$$3x_0 = x_0\,e^{k(5)}$$

$$3 = e^{5k}$$

$$3 = \left(e^k\right)^5$$

$$\boxed{e^k = 3^{\frac{1}{5}}}$$

$$\text{Sub } e^k = 3^{\frac{1}{5}} \text{ in } ②$$

$$x = x_0\left(e^k\right)^t$$

$$x = x_0\left(3^{\frac{1}{5}}\right)^t$$

$$x = x_0\,3^{\frac{t}{5}} \rightarrow ③$$

$$\text{When } t = 10,\ x = ?$$

$$x = x_0\,3^{\frac{10}{5}}$$

$$x = 9x_0$$

$\therefore$ The no. of bacteria after 10 hours is 9 times of initial no. of bacteria.

(OR)

(b) Find the vector and Cartesian equations of the plane containing $\frac{x-2}{2}=\frac{y-2}{3}=\frac{z-1}{1}$ and parallel to the line $\frac{x+1}{3}=\frac{y-1}{2}=\frac{z+1}{1}$.

Solution:

The requied equation of plane passing through a point $(2, 2, 1)$ and parallel to vectors $2\hat{i}+3\hat{j}+\hat{k}$ and $3\hat{i}+2\hat{j}+\hat{k}$

Let $\vec{a}=2\hat{i}+2\hat{j}+\hat{k}$; $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$; $\vec{c}=3\hat{i}+2\hat{j}+\hat{k}$

Vector equation:

$$\vec{r}=\vec{a}+s\vec{b}+t\vec{c} \quad \text{where } s,t\in\mathbb{R}$$

$$\vec{r}=(2\hat{i}+2\hat{j}+\hat{k})+s(2\hat{i}+3\hat{j}+\hat{k})+t(3\hat{i}+2\hat{j}+\hat{k}) \quad \text{where } s,t\in\mathbb{R}$$

Cartesian equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$(x_1,y_1,z_1)=(2,2,1)$$

$$(b_1,b_2,b_3)=(2,3,1)$$

$$(c_1,c_2,c_3)=(3,2,1)$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix}=0$$

$$(x-2)(3-2)-(y-2)(2-3)+(z-1)(4-9)=0$$

$$1(x-2)+1(y-2)-5(z-1)=0$$

$$x-2+y-2-5z+5=0$$

$$\boxed{x+y-5z+1=0}$$

Non − parametric vector equation:

$$\boxed{\vec{r}.(-\hat{i}-\hat{j}+5\hat{k})=1}$$

46. (a) Find the area of the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

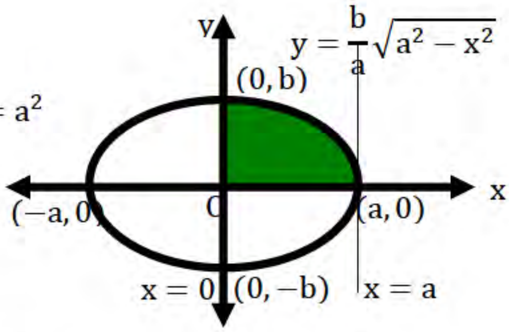
$$x-\text{axis} \Rightarrow y=0 \Rightarrow \frac{x^2}{a^2}+0=1 \Rightarrow x^2=a^2$$

$$x=\pm a$$

The limits are $x=0$ and $x=a$

The required area is

$$A=4\int_0^a y\,dx$$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \Rightarrow y^2=\frac{b^2}{a^2}(a^2-x^2) \Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}$$

$$A=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=\frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$=\frac{4b}{a}\left[\left(0+\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right)-0\right]$$

$$=\frac{4b}{a}\left[\frac{a^2}{2}\left(\frac{\pi}{2}\right)\right]$$

$$=\pi ab$$

(OR)

(b) Find the vertex, focus, equation of directrix of the parabola $y^2 - 4y - 8x + 12 = 0$.

Solution:

$$\text{G.T.}\quad y^2 - 4y - 8x + 12 = 0$$
$$y^2 - 4y = 8x - 12$$
$$y^2 - 4y + 4 = 8x - 12 + 4$$
$$y^2 - 4y + 4 = 8x - 8$$
$$(y-2)^2 = 8(x-1) \text{ it is open right}$$
$$\text{Here } h = 1, k = 2, 4a = 8 \Longrightarrow a = 2$$

Vertex : $A(h,k) = A(1,2)$

Focus : $S(h+a,k) = S(1+2,2) = S(3,2)$

Equ. of directrix : $x = h - a \Longrightarrow x = 1 - 2 \Longrightarrow x + 1 = 0$

47. (a) Show that $p \leftrightarrow q \equiv ((\sim p) \vee q) \wedge ((\sim q) \vee p)$.

Solution

p	q	$p \leftrightarrow q$	$\sim p$	$\sim q$	$(\sim p) \vee q$	$(\sim q) \vee p$	$((\sim p) \vee q) \wedge ((\sim q) \vee p)$
T	T	T	F	F	T	T	T
T	F	F	F	T	F	T	F
F	T	F	T	F	T	F	F
F	F	T ①	T	T	T	T	T ②

From ① and ②, we get

$$p \leftrightarrow q \equiv ((\sim p) \vee q) \wedge ((\sim q) \vee p)$$

(OR)

(b) Solve the system of linear equations by Cramer's Rule.

$\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0,\ \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0.$

Solution:

$$\text{Let}\quad a = \frac{1}{x},\quad b = \frac{1}{y},\quad c = \frac{1}{z}$$
$$3a - 4b - 2c = 1$$
$$a + 2b + c = 2$$
$$2a - 5b - 4c = -1$$

$$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = 3(-8+5) + 4(-4-2) - 2(-5-4) = -9 - 24 + 18 = -15 \neq 0$$

$$\Delta_a = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix} = 1(-8+5) + 4(-8+1) - 2(-10+2) = -3 - 28 + 16 = -15$$

$$\Delta_b = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix} = 3(-8+1) - 1(-4-2) - 2(-1-4) = -21 + 6 + 10 = -5$$

$$\Delta_c = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix} = 3(-2+10) + 4(-1-4) + 1(-5-4) = 24 - 20 - 9 = -5$$

By Crammer's rule,

$$a = \frac{\Delta_a}{\Delta} = \frac{-15}{-15} = 1 \quad \Longrightarrow \frac{1}{x} = 1 \Longrightarrow \boxed{x = 1}$$
$$b = \frac{\Delta_b}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \quad \Longrightarrow \frac{1}{y} = \frac{1}{3} \Longrightarrow \boxed{y = 3}$$
$$c = \frac{\Delta_c}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \quad \Longrightarrow \frac{1}{z} = \frac{1}{3} \Longrightarrow \boxed{z = 3}$$

Hence The solution is $x = 1,\ y = 3,\ z = 3$