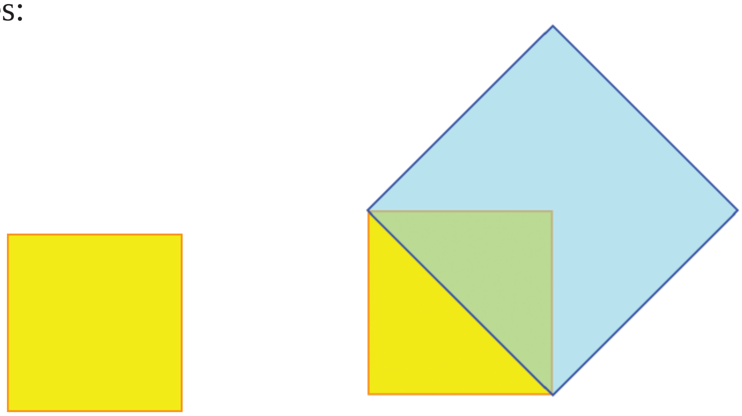
2

New Numbers

Lengths and numbers

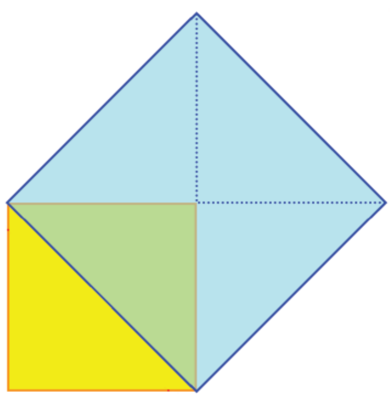
See these pictures:



A square, and on its diagonal another square.

How many times the area of the small square is the area of the large square ?

The diagonal splits the small square into two equal right triangles. How many such right triangles make up the big square ?



So, the area of the big square is twice the area of the small square.

If the side of the small square is 1 metre, then its area is 1 square metre; and the area of the big square is 2 square metre.

What is the length of a side of the big square?

Since it is the diagonal of the small square, it is larger than 1 metre.

Since it is the third side of a triangle of other two sides 1 metre each, it is less than 2 metres.

The length can be a fraction between 1 and 2. But then its square must be 2, being the side of a square of area 2 square metres.

What fraction has its square equal to 2 ?

Can it be one and a half ?

$$\left(1\frac{1}{2}\right)^2 = 1 + 1 + \frac{1}{4} = 2\frac{1}{4}$$

It's a bit too large. How about one and a quarter?

$$\left(1\frac{1}{4}\right)^2 = 1 + \frac{1}{2} + \frac{1}{16} = 1\frac{9}{16}$$

A bit too small now.

What about one and a third?

$$\left(1\frac{1}{3}\right)^2 = 1 + \frac{2}{3} + \frac{1}{9} = 1\frac{7}{9}$$

That too is smaller than what we seek, but better than one and a quarter.

If we go on trying different fractions, the squares get closer and closer to 2, but never exactly 2. In fact, we can prove using algebra that it isn't possible (See the appendix at the end of the lesson).

In other words,

The square of any fraction is not equal to 2

We have a real problem here:

On one hand, if the length of the diagonal of a square of side 1 metre is a fractional multiple of 1 metre, then the square of that fraction must be 2 (We have seen that even if the side of a square is a fraction, its area is the square of the fraction.)

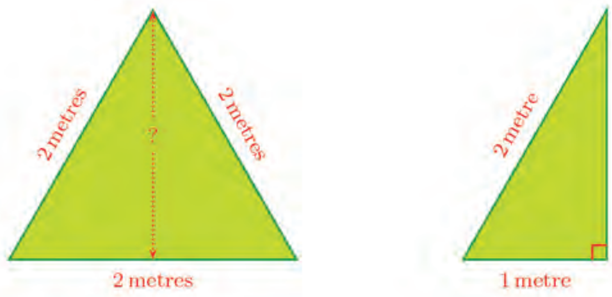
On the other hand, there is no fraction whose square is 2.

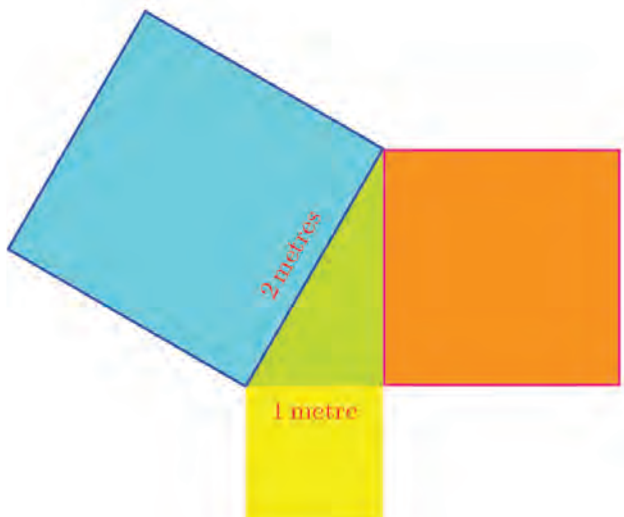
Thus we are led to this conclusion:

The length of the diagonal of a square of side 1 unit cannot be expressed as a fraction.

There are other lengths like this, which cannot be expressed as natural numbers or fractions. For example, let's compute the height of an equilateral triangle of sides 2 metres.

If we cut such a triangle into two along its height and take one piece, we get a right triangle:



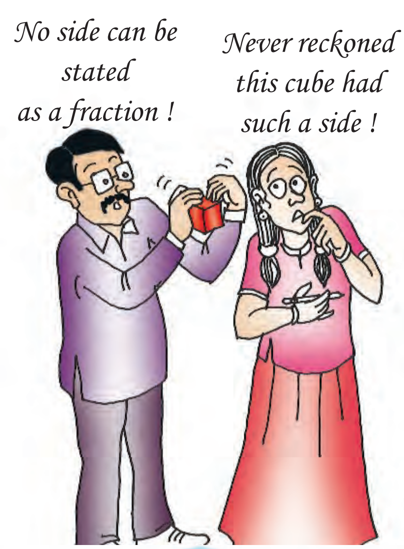


To compute the length of its third side, let's draw squares on all sides

By the Pythagoras Theorem, the area of the orange square is $4 - 1 = 3$ square metres. So, if its side is a fractional multiple of 1 metre, then the square of this fraction must be 3.

Just as we show that the square of no fraction is 2, we can also show that the square of no fraction is 3. So, the height of this triangle cannot be expressed as a fraction.

Let's look at another example. Suppose we want to make a cube of volume 2 cubic centimeters. What should be the length of a side? Just as the square of no fraction is equal to 2, the cube of no fraction is equal to 2 either. So, we cannot express the length of a side of this cube as a fraction.



In many such contexts, we need lengths which cannot be expressed as fractions.

Number evolution

Convert everything to numbers and try to make sense of the world through such numbers and their interrelations, this is one of the primary concerns of mathematics.

Depending on the nature of the things measured, different kinds of numbers have to be invented. During the period when humans were just hunter gatherers, they needed only such numbers as the number of people in a group or the number of cattle they owned. At that time only natural numbers was necessary.

Around BCE 5000 they began to settle down by the side of the great rivers and started extensive agriculture. Then they had to measure lengths and areas to mark farmlands and to build houses. Fractions were invented at this stage. Fractions are also necessary for fair division. New numbers became necessary with the realization that not all measurements can be indicated by fractions.

Later, new numbers were created not only for physical needs, but also as mathematical conveniences. Negative numbers and complex numbers were invented for such a purpose. That such numbers also were found to be useful in physical sciences is another side of the story.

Measures and numbers

To indicate lengths which cannot be expressed as natural numbers or fractions, we have to create new numbers. Let's have a look at our first example. How do we represent the length of the diagonal of a square of side 1 metre ?

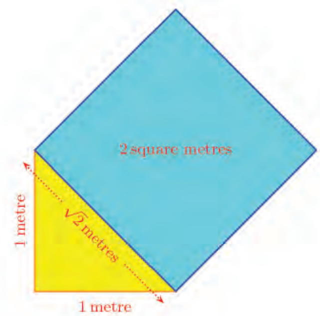
Let's put it this way: how do we indicate the side of a square of area 2 square metres?

If the side of a square is a natural number or fraction, the length of a side is the square root of the area.

For example, the side of a square of area 4 square metres is $\sqrt{4} = 2$ metres; if the area is $2\frac{1}{4}$ square metres, the length of a side is $\sqrt{2\frac{1}{4}} = 1\frac{1}{2}$ metres.

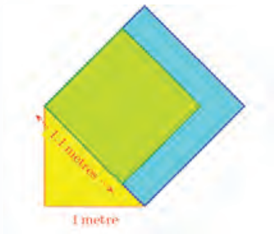
In the same way, we write

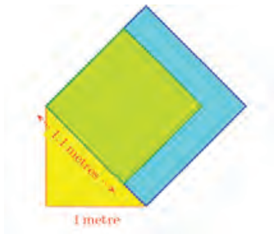
The side of a square of area 2 square metres is $\sqrt{2}$ metres

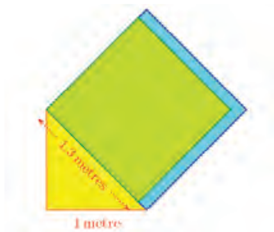


Just giving a symbol to denote a length isn't enough; to know its size we must be able to compare it with known lengths.

To do it, we must find lengths which can be expressed as fractions and which get closer and closer to this length. If such lengths are marked on the diagonal itself, the squares on these lengths get closer and closer to the square on the diagonal.







In terms of just numbers, this means the squares of the fractions giving these lengths get nearer and nearer to 2. We can find these squares using a calculator:

$$1.1^2 = 1.21$$

$$1.2^2 = 1.44$$

$$1.3^2 = 1.69$$

$$1.4^2 = 1.96$$

$$\mathbf{1.5^2 = 2.25}$$

What do we see here?

$$1.4^2 < 2 < 1.5^2$$

Let's explain this method of finding fractions whose squares get closer and closer to 2.

Considering only natural numbers, we get,

$$1^2 < 2 < 2^2$$

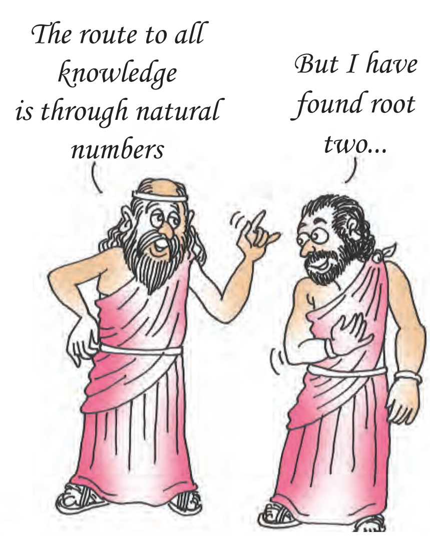
Collapsing beliefs

Pythagoras, who lived around the sixth century BCE, and his followers believed that all measurements could be compared using just natural numbers. More precisely, that any two measurements can be represented together as a ratio of natural numbers.

But the ratio of the lengths of the diagonal and side of a square cannot be thus represented. For if this ratio can be given as $a : b$, where a and b are natural numbers, then the diagonal must be $\frac{a}{b}$ times the side, which means the square of the length of the diagonal must be $\left(\frac{a}{b}\right)^2$ times the square of the length of the side; since the area of the square on the diagonal is twice the area of the square on the side, this would mean, $\left(\frac{a}{b}\right)^2 = 2$, which we have seen is impossible.

It is believed that this was first discovered by Hippasus, himself a disciple of Pythagoras.

Pairs of lengths, such as the diagonal and side of a square, which cannot be compared using a ratio of natural numbers are called incommensurable magnitudes.



Suppose we take tenths also into our reckoning

$$\left(1\frac{4}{10}\right)^2 < 2 < \left(1\frac{5}{10}\right)^2$$

What if we take hundredths also?

$$1.41^2 = 1.9881$$
$$1.42^2 = 2.0164$$

Thus

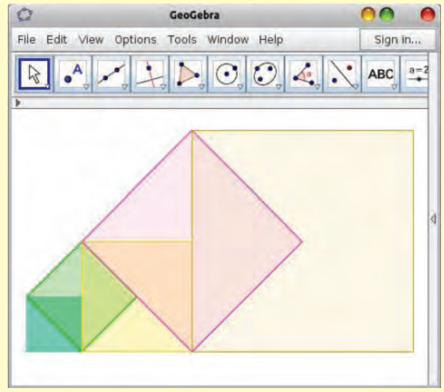
$$1.41^2 < 2 < 1.42^2$$

or in other words

$$\left(1\frac{41}{100}\right)^2 < 2 < \left(1\frac{42}{100}\right)^2$$

Continuing like this, we can see

$1.4^2 = 1.96$	$1.5^2 = 2.25$
$1.41^2 = 1.9881$	$1.42^2 = 2.0164$
$1.414^2 = 1.999396$	$1.415^2 = 2.002225$
$1.4142^2 = 1.99996164$	$1.4143^2 = 2.00024449$
$1.41421^2 = 1.9999899241$	$1.41422^2 = 2.0000182084$



In the picture, the side of the smallest square is 1 centimetre. Calculate the area and the side of the largest square. Draw this in GeoGebra (Use the Regular Polygon tool). Use Area tool to find the area of each square. The sides of which of these squares can be expressed as fractions ?

That is if we take up to the fifth decimal place (up to hundred thousandths),

$$1.41421^2 < 2 < 1.41422^2$$

In short,

The squares of the fractions $1\frac{4}{10}$, $1\frac{41}{100}$, $1\frac{414}{1000}$, $1\frac{4142}{10000}$, $1\frac{41421}{100000}$ and so on get nearer and nearer to 2.

In terms of decimals,

The squares of the fractions 1.4, 1.41, 1.414, 1.4142, 1.41421 and so on get nearer and nearer to 2.

This we write in shortened form like this:

$$\sqrt{2} = 1.41421...$$

We also say that the number $\sqrt{2}$ is 1.4 up to one decimal place, 1.41 up to two decimal places and so on.

Nearer and Nearer

$2 - 1.4^2 = 0.04$

$2 - 1.41^2 = 0.0119$

$2 - 1.414^2 = 0.000604$

$2 - 1.4142^2 = 0.00003836$

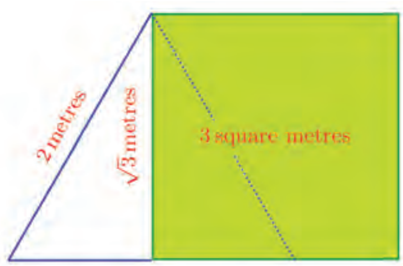
$2 - 1.41421^2 = 0.0000100759$

These are written,

$$\sqrt{2} \approx 1.4$$
$$\sqrt{2} \approx 1.41$$

Here the symbol ≈ is to be read "approximately equal to".

Similarly, since the area of the square drawn on the height of an equilateral triangle of side 2 metres is 3 square meters (as we have seen earlier), we can denote this height by $\sqrt{3}$



By computations as done earlier, we can find that the squares of the fractions 1.7, 1.73, 1.732 and so on get nearer and nearer to 3. This also we shorten as

$$\sqrt{3} = 1.73205...$$

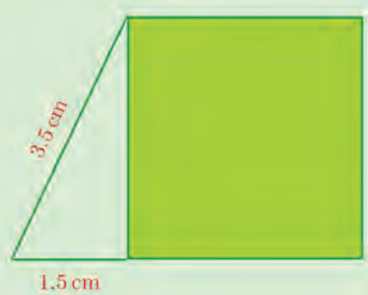
Generally speaking,

$\sqrt{x}$ is the side of a square of area x, for any positive number x

In some cases, $\sqrt{x}$ may be a natural number or a fraction. If not, we can compute fractions whose squares get nearer and nearer to x and write $\sqrt{x}$ in decimal form.

(1) We have seen in class 8 that any odd number can be written as the difference of two perfect squares. Use this to draw squares of area 7 square centimetres and 11 square centimetres. What are the lengths of the sides of these squares?

(2) What is the area of the square in the picture? What is the length of its sides?



Line and root

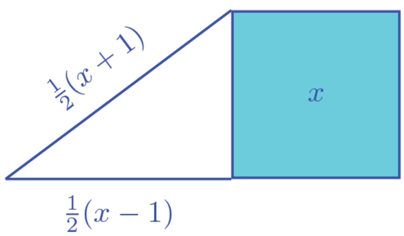
For any number x, we have

$$(x+1)^2 - (x-1)^2 = 4x$$

(The lesson **Square Equations** in Class 8) This we can rewrite as

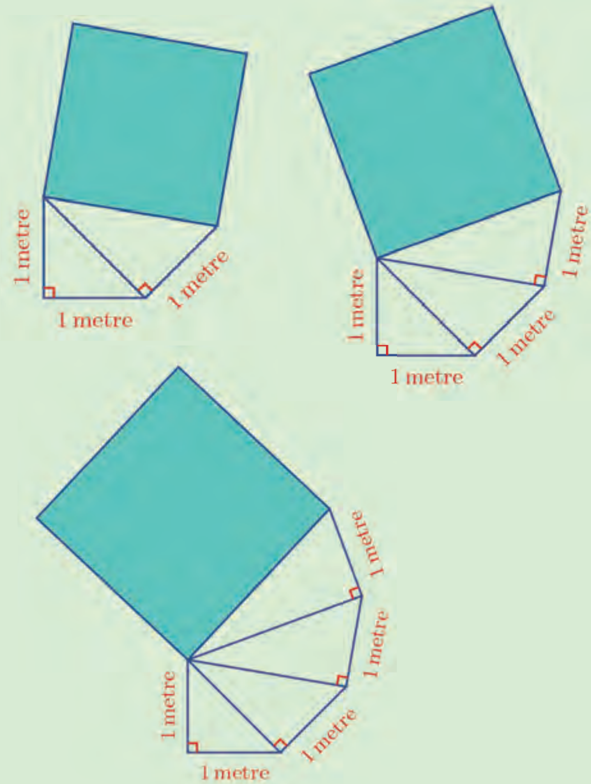
$$x = \left(\tfrac{1}{2}(x+1)\right)^2 - \left(\tfrac{1}{2}(x-1)\right)^2$$

and use this to draw a square of area x for any x. For $x > 1$, this is how we do it:



For $x < 1$, we take the length of the base of triangle as $\frac{1}{2}(1-x)$

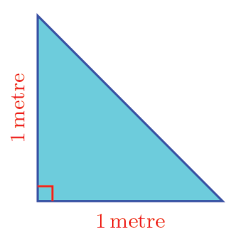
(3) Calculate the area of the square and the length of its sides in each of the pictures below:



(4) Find three fractions greater than $\sqrt{2}$ and less than $\sqrt{3}$

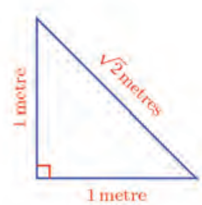
Addition and subtraction

What is the area of a right triangle of perpendicular sides 1 metre each ?



And the perimeter?

We know that its hypotenuse is $\sqrt{2}$ metres



So to calculate the perimeter, we have to add 2 metres and $\sqrt{2}$ metres.

We write this length as $2+\sqrt{2}$.

The fractions 1.4, 1.41, 1.414, 1.4142 and so on are approximate values of $\sqrt{2}$.

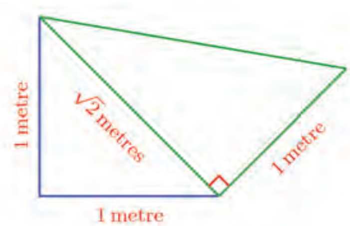
So, by adding 2 to each of these, we get fractions approximating $2+\sqrt{2}$; that is, the fractions, 3.4, 3.41, 3.414, 3.4142 and so on.

This we write as

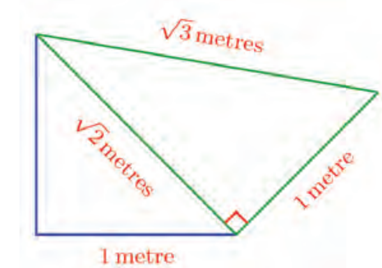
$$2+\sqrt{2} = 3.4142...$$

If we decide to have accuracy only up to a centimetre, we can take the perimeter as 3.41 metres; instead if we need accuracy up to a millimetre, we take the perimeter as 3.414 metres.

Now suppose we draw another triangle with the hypotenuse of the first as its base, as below:



We've seen that the third side of this triangle is $\sqrt{3}$ metres.



Decimal forms

Decimals were first used as a shorthand for fractions with powers of 10 as denominators, such as

$$\frac{7}{10} = 0.7, \qquad \frac{21}{100} = 0.21$$

$$\frac{1}{2} = \frac{5}{10} = 0.5, \qquad \frac{1}{4} = \frac{25}{100} = 0.25$$

Later this was extended to other fractions, in a different form. For example, since the fractions

$$\frac{3}{10}, \frac{33}{100}, \frac{333}{1000}, \ldots$$

get closer and closer to $\frac{1}{3}$, we write

$$\frac{1}{3} = 0.333...$$

Similarly, we can compute

$$\frac{1}{6} = 0.1666...$$

$$\frac{1}{11} = 0.090909...$$

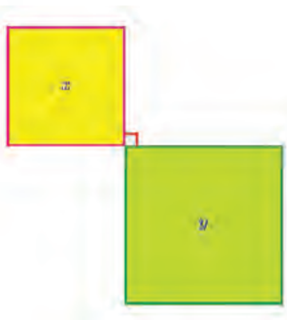
In the same way, we write

$$\sqrt{2} = 1.41421...$$

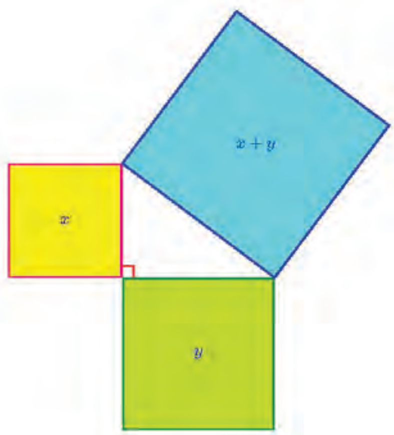
But there's a difference. In the decimal forms of fractions such as $\frac{1}{3}$, $\frac{1}{6}$ or $\frac{1}{11}$ we see groups of digits repeating again and again. In the decimal forms of numbers like $\sqrt{2}$ or $\sqrt{3}$, there are no such repetitions.

Sum and root

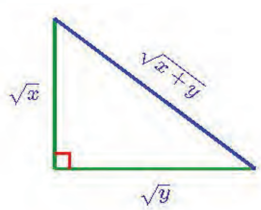
Place two squares of areas x and y like this:



Next let's join the top corrners and draw a square on it:



What are the lengths of the sides of triangle in the middle?



What do get from this?

$$\sqrt{x}+\sqrt{y}>\sqrt{x+y}$$

So the perimeter of this triangle is $1+\sqrt{2}+\sqrt{3}$. To get factions approximating $\sqrt{2}+\sqrt{3}$, we must add the approximations to each of these in order:

$$\begin{array}{lllll} 1.4 & 1.41 & 1.414 & \dots & \rightarrow \sqrt{2} \\ 1.7 & 1.73 & 1.732 & \dots & \rightarrow \sqrt{3} \\ \hline 3.1 & 3.14 & 3.146 & \dots & \rightarrow \sqrt{2}+\sqrt{3} \end{array}$$

Adding 1 to each of these give approximations to $1+\sqrt{2}+\sqrt{3}$

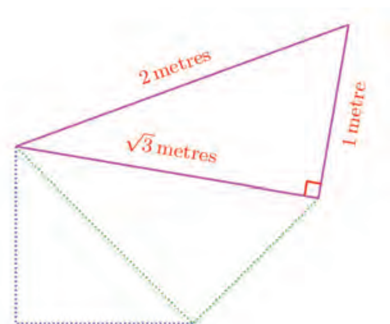
Thus the perimeter of the new triangle is 4.146 metres, correct to a millimetre

How much larger is the perimeter of this triangle than that of the first one ?

We can say approximately 4.146 – 3.414 = 0.732 metres. Or we can compute like this:

$$\begin{aligned}(1+\sqrt{2}+\sqrt{3})-(2+\sqrt{2}) &= 1+\sqrt{3}-2 \\ &= \sqrt{3}-1 \\ &\approx 0.732\end{aligned}$$

Now suppose we draw yet another triangle on top of this as below: What are the lengths of the sides ?



How much larger is the perimeter of this than that of the second one?

The perimeter of the new triangle is

$$2 + 1 + \sqrt{3} = 3 + \sqrt{3}$$

Let's compute how much larger is the perimeter, without actually calculating its approximate values.

What is the perimeter of the second triangle? So, the difference in perimeters is

$$(3 + \sqrt{3}) - (1 + \sqrt{2} + \sqrt{3}) = 2 - \sqrt{2}$$

We can calculate this upto three decimals as

$$2 - 1.414 = 0.586$$

That is, the perimeter is about 586 millimetres or 58.6 centimetres more

Subtraction

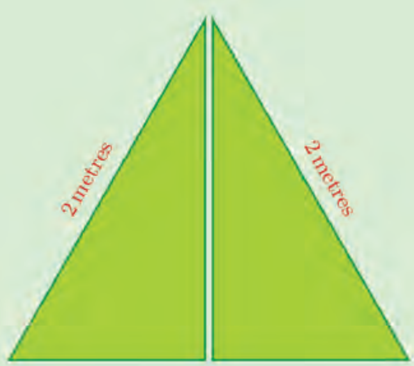
We calculate fractions approximating $\sqrt{3} - \sqrt{2}$, just as we did for $\sqrt{3} + \sqrt{2}$.

1.7	1.73	1.732	...	$\rightarrow$	$\sqrt{3}$
1.4	1.41	1.414	...	$\rightarrow$	$\sqrt{2}$
0.3	0.32	0.318	...	$\rightarrow$	$\sqrt{3} - \sqrt{2}$

$$\sqrt{3} - \sqrt{2} \approx 0.318$$

(1) The hypotenuse of a right triangle is $1\frac{1}{2}$ metres and one of the other sides is $\frac{1}{2}$ metre. Calculate its perimeter, up to a centimetre.

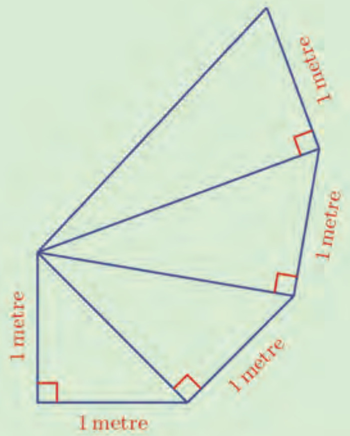
(2) The picture below shows an equilateral triangle cut into two triangles along a line through the middle:

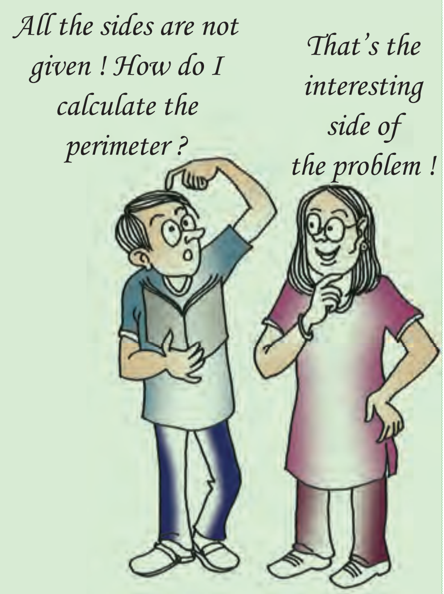


(i) What is the perimeter of one of these ?

(ii) How much is it less than the perimeter of the whole triangle ?

(3) We've seen how we can go on drawing right triangles like this:





(i) What are the lengths of the sides of the tenth triangle in this pattern?

(ii) How much more is the perimeter of the tenth triangle than that of the ninth?

(4) What is the hypotenuse of a right triangle with perpendicular sides $\sqrt{2}$ centimetres and $\sqrt{3}$ centimetres ? How much more is the sum of the perpendicular sides than the hypotenuse ?

Appendix

Let's see how we can prove that there is no fraction whose square is 2.

We start by looking at the specialities that the numerator and denominator of such a fraction. We know that every fraction has various forms and among these there's one in its lowest terms, that is, a form in which the numerator and denominator have no common factor. Let's take this form of the fraction we are seeking, the one whose square is 2, as $\frac{x}{y}$. So, x and y are natural numbers with no common factor.

What other properties do they have?

$$\left(\frac{x}{y}\right)^2 = 2$$

That is,

$$\frac{x^2}{y^2} = 2$$

which gives

$$x^2 = 2y^2$$

So, x^2 is an even number. What about x itself ?

The squares of odd numbers are all odd numbers (and the squares of even numbers, even). Since x^2 is even, so must be x itself.

Now x is an even number and x and y have no common factors. So, y cannot be an even number (2 is a common factor of any two even numbers, right ?). This means y is an odd number. Thus in the fraction we are seeking, the numerator is even and the denominator is odd. Can we find anything more ? Let's continue our investigation.

Since x is an even number, we can divide it by 2 to get a natural number. That is $\frac{x}{2}$ is a natural number. Let's write it as z:

$$\frac{x}{2} = z$$

That is,

$$x = 2z$$

Now we can write $2z$ in the place of x in our first equation $x^2 = 2y^2$:

$$(2z)^2 = 2y^2$$

That is

$$4z^2 = 2y^2$$

From this we get

$$y^2 = 2z^2$$

This means y^2 is an even number. From this we can see, as before in the case of x, that y itself is even.

But how can this be ? We've already noted that y is an odd number.

What happened here ?

For a fraction in the lowest terms whose square is 2, we first saw that the denominator is an odd number; further analysis showed the denominator must be an even number. And we cannot have both.

Thus we see there is no fraction whose square is 2.