# 3

# PARALLEL LINES

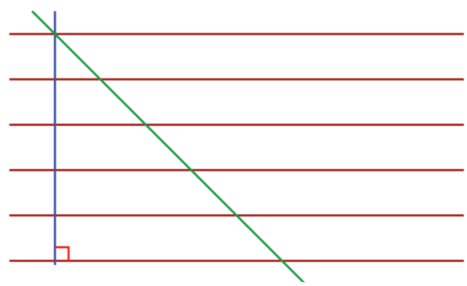
## Equal division

We've learnt many things about parallel lines, and drew many a figure using these. There's much more.

See this picture:

A bunch of parallel lines and a line perpendicular to them all.

The parallel lines are drawn the same distance apart. In other words, the parallel lines cuts the perpendicular into equal pieces.



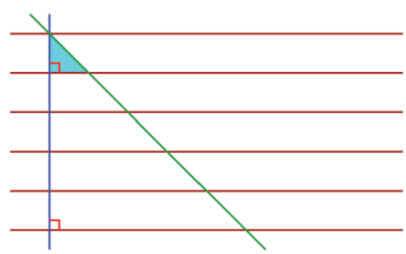
What if we draw a slightly slanted line instead of a perpendicular ?

Doesn't it look as if the parallel lines cut

this also into equal pieces? How do we check?

We can measure them;

The fun in math lies in finding things by reasoning than by actually doing.



Do you see a small right triangle at the top of the picture?

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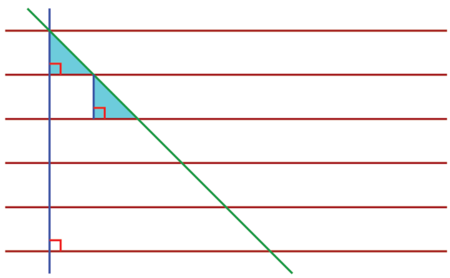
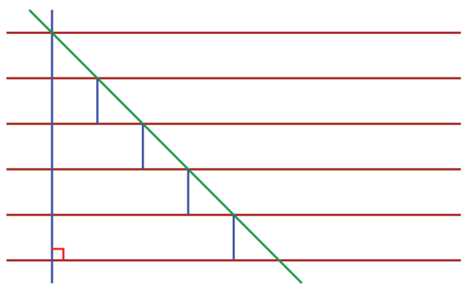
Its vertical side is a piece of the perpendicular; and the hypotenuse is a piece of the slanted line.

If we draw one more small perpendicular, we get another triangle just below the first:



The vertical (blue) sides of these triangles are equal (why?) Also, since these perpendiculars are parallel, the slanted (green) line is equally inclined to both.

In other words, the vertical sides of these triangles are equal and so are its angles on either side of these lines. So, their hypotenuses must also be equal, right?



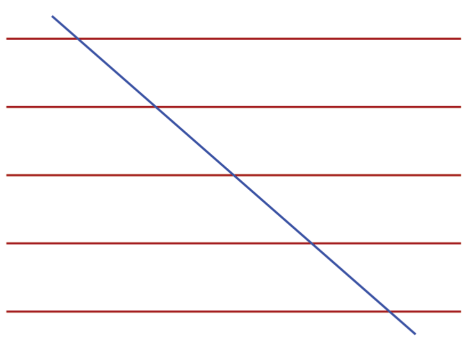
Thus the top two pieces of the slanted line are of equal length.

In the same way, we can see that the other pieces are all equal.

We state this as a general result:

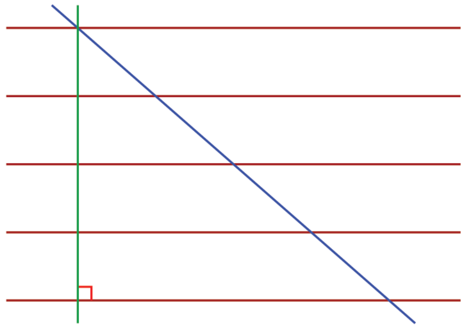
**Parallel lines the same distance apart cut any other line into equal pieces.**

What about the other way round? That is, suppose that some parallel lines cut a slanted line into equal pieces. Can we say that they are the same distance apart ?

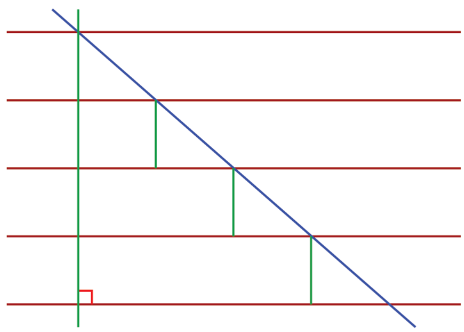


The parallel lines in the picture cut the slanted blue line into equal pieces. The question is whether the distances between adjacent pairs the same.

In other words, we must check whether the parallel lines cut the green perpendicular also into equal pieces.



Let's draw small perpendiculars as before:

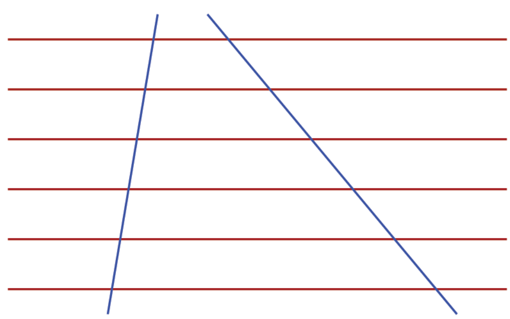


As in the first case, all the little triangles have the same angles. And their hypotenuses are also equal. So, their vertical sides are also equal.

Thus the parallel lines are the same distance apart.

We state this also as a general result:

**Parallel lines which cut some line into equal pieces are equal distance apart.**



Using these two results, we can see another thing. Suppose a bunch of parallel lines cut some line into equal pieces. By the result just seen, they are equal distance apart.

So, by the first result, they cut any other line also into equal pieces.

Thus we have the result below:

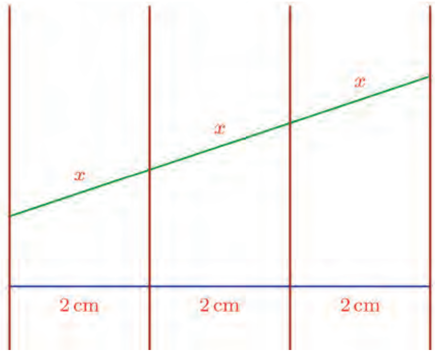
Parallel lines which cut one line into equal pieces, cut all lines into equal pieces

We can use this to cut any line into any number of equal pieces.

For example, let's see how we can divide a 7 centimetre long line into three equal pieces.

This is easy for a 6 centimetre long line.

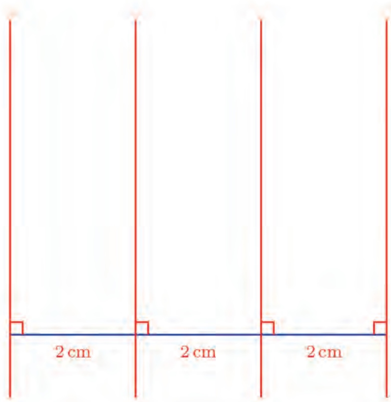
Four parallel lines which divide a 6 centimetre long line into three equal parts, divide any other line into three equal pieces, don't they?

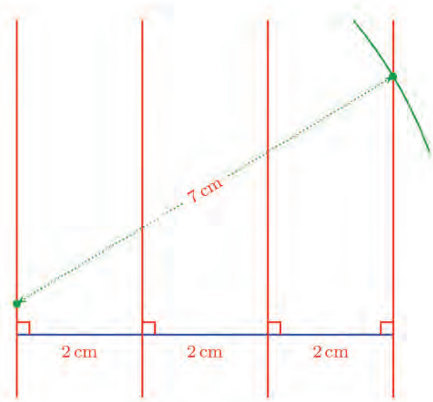



If we make the other line 7 centimetre long ?

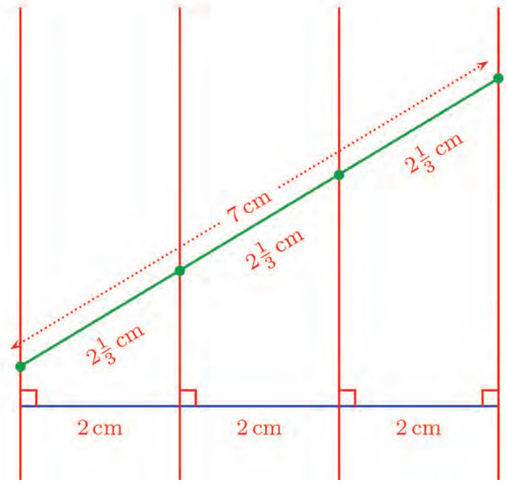
Now the way is clear to solve our problem.

Draw a line 6 centimetre long and draw on it, perpendiculars 2 centimetres apart. From any point on the first perpendicular, draw an arc of radius 7 centimetres and mark the point where it cuts the last perpendicular:

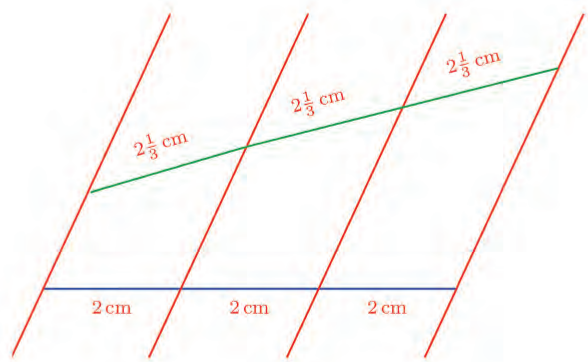


Joining these two points, we get a 7 centimetre long line cut into three equal pieces.



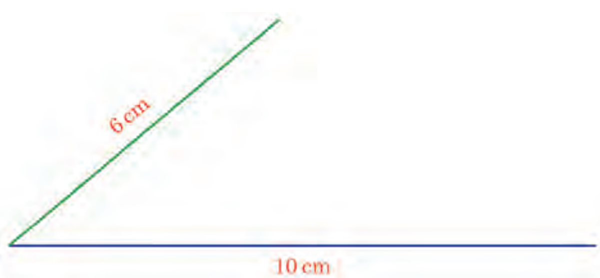

It is not really necessary that we should draw perpendiculars to the first line. Slanted lines also will do; but they must be parallel:



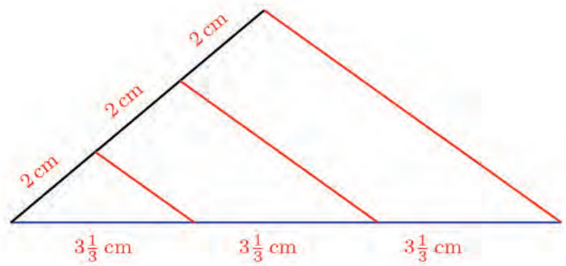

Another problem:

Draw a line 10 centimetre long and divide it into three equal pieces.

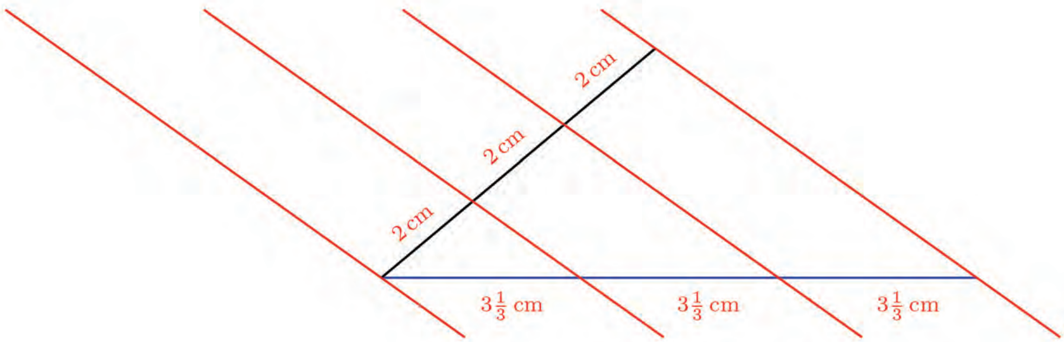
It is more convenient to draw horizontally the line to be divided. Then we can draw the line which helps the cutting, slanted from one end of the first:



Now draw a line joining the other ends of the two lines. Next we need only divide the top line into three equal parts and draw parallels through these points:

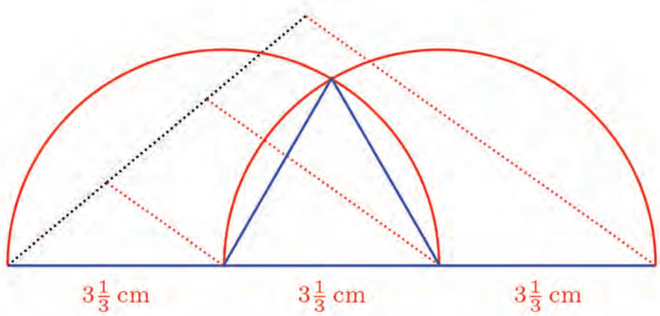



If this is not clear, extend these parallel line a little and draw a fourth parallel:




Thus the 10 centimetre long line is cut into three equal parts.

How about drawing a triangle joining these pieces?




This is an equilateral triangle, since all three sides are of the same length. And their sum is 10 centimetres. That is, the perimeter of the triangle is 10 centimetres.

Now suppose we want to divide a 10 centimetre long line into four equal parts instead of three equal parts. What is a convenient length for the slanted line at the top?

And after doing such a division, we can also draw a square of perimeter 10 centimetres. Try it.

And also these:

(1) Draw an equilateral triangle of perimeter 11 centimetres.

(2) Draw a square of perimeter 15 centimetres.

(3) Draw a regular hexagon of perimeter 20 centimeters.

## Unequal division

See this picture:

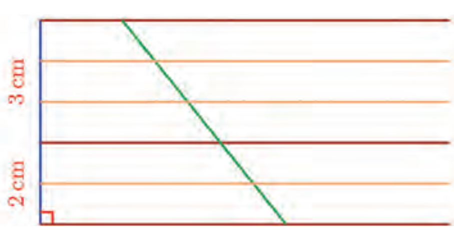

Three parallel lines with different distances between them

Let's draw a slanted line as before:


We can tell at a glance the lengths of the pieces of the green line are not 2 centimetres and 3 centimetres.

Do these lengths have any relation with 2 and 3 ?

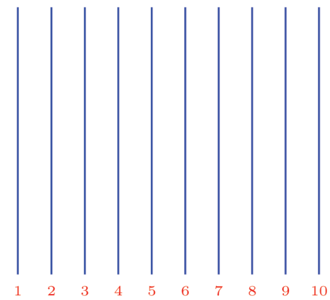
Suppose we draw three more parallel lines to make six equally spaced parallel lines ?



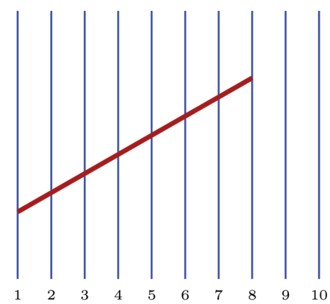

Now the slanted line is split into five equal parts. The top three pieces make up the longer of the two pieces we saw first; and the bottom two make up the shorter piece.

## Parallel division

Draw some equally spaced parallel lines on a sheet of paper:

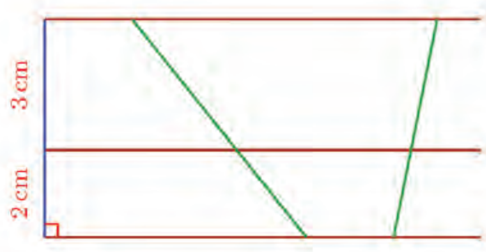



Now to divide a thin wire into seven equal pieces, all we need to do is to place it across eight of these line as shown below:




In other words, the shorter and longer parts of the slanted line are in the ratio 2 : 3.

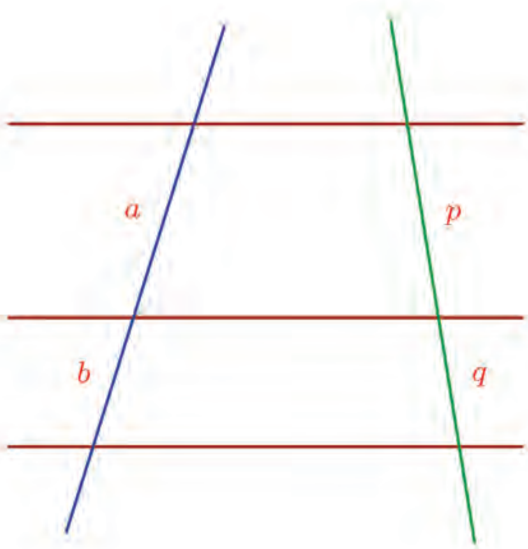
What if we draw another line ?





The actual lengths of the parts are different, but still two of the equal parts make up the shorter part and three of them, the longer part. That is, the ratio is still 2 : 3

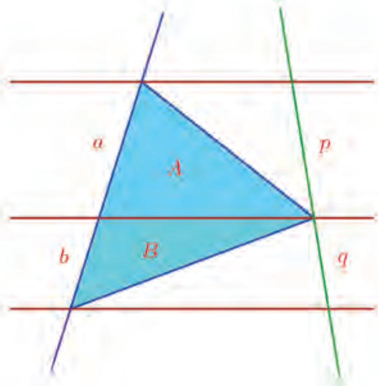
Do any three parallel lines divide all other lines in the same ratio?




In the picture above, there are three horizontal lines which are parallel and two slanted lines which cut through them. Let's denote the lengths of the pieces of the left line as $a$, $b$ and those of the right line as $p$, $q$.

We want to check whether the ratios $a : b$ and $p : q$ are the same. In other words, the question is whether $\frac{a}{b} = \frac{p}{q}$.

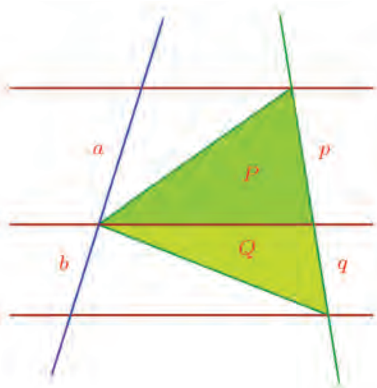
To check this, we first change the ratio of lengths to ratio of areas:




We know that the ratio of the areas of the top and bottom blue triangles is also $a : b$. That is, if the areas of these triangles are denoted as $A$ and $B$, then

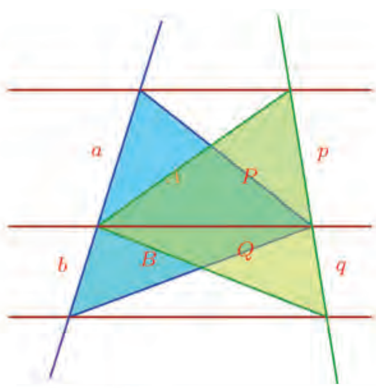
$$\frac{a}{b} = \frac{A}{B}$$

Similarly, we can also convert the ratio of the lengths $p$ and $q$ also to a ratio of areas:

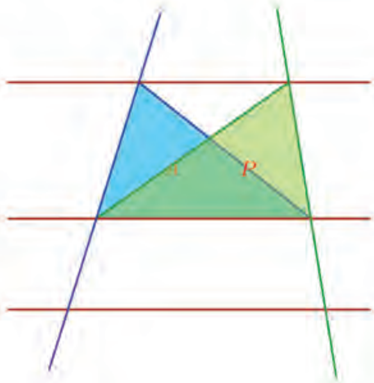


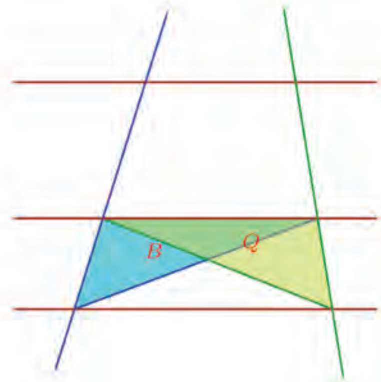

If we denote the areas of the green triangles as $P$ and $Q$ as in the picture above, $\frac{p}{q} = \frac{P}{Q}$

Now let's put all the triangles together:



There is a blue triangle and green triangle each at the top and bottom. Let’s look at them pair by pair:




Both triangles at the top have the same bottom side; and their top corners are on a line parallel to this side. So, they have the same area.

$$A = P$$

Same is the case with the blue and green triangles at the bottom.

$$B = Q$$

We have already seen that

$$\frac{a}{b} = \frac{A}{B} \text{ and } \frac{p}{q} = \frac{P}{Q}$$

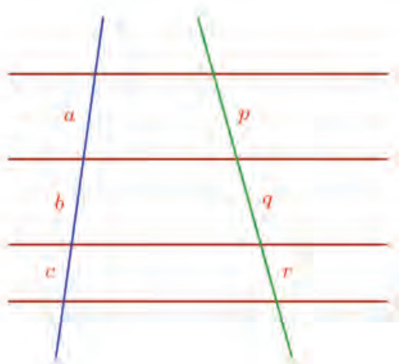
And now we see that $A = P$ and $B = Q$. Thus,

$$\frac{a}{b} = \frac{p}{q}$$

In other words, the three parallel lines divide the left and right lines in the same ratio.

What if there are more than three lines ?

See this picture:



If we just look at the top three lines only, we get $\frac{a}{b} = \frac{p}{q}$, as we did just now.

Instead if we look at the bottom three lines only, we get $\frac{b}{c} = \frac{q}{r}$.

From $\frac{a}{b} = \frac{p}{q}$ and $\frac{b}{c} = \frac{q}{r}$, we get $\frac{a}{b} \times \frac{b}{c} = \frac{p}{q} \times \frac{q}{r}$. That is, $\frac{a}{c} = \frac{p}{r}$.

So among the lengths *a*, *b*, *c* whatever part or times of one is another, the same part or times is the corresponding pieces of *p*, *q*, *r*

Thus the ratio between the lengths *a*, *b*, *c* is same as the ratio between the lengths *p*, *q*, *r*.

We can continue like this for any number of parallel lines.

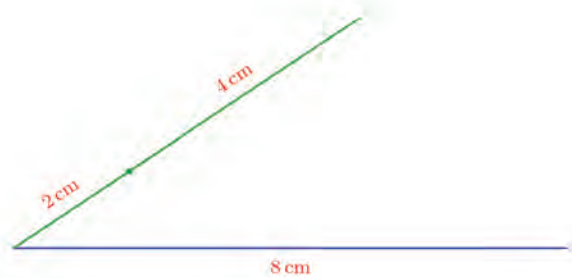
Three or more parallel lines cut any two lines in the same ratio.

We can use this to divide a line in any ratio.

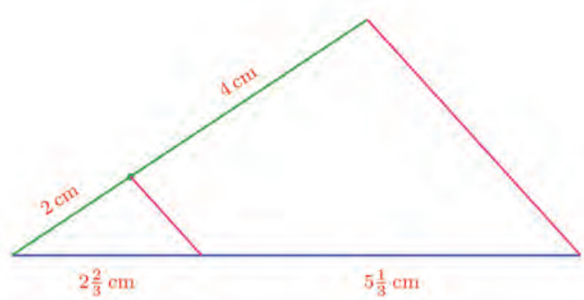
For example, let's see how we divide an 8 centimetre long line in the ratio 1 : 2.

Isn't it easy to divide a 6 centimetre line in this ratio ?

So let's first draw two lines as we did earlier:




Now we need only join the other ends of the lines and draw the parallel line through the point dividing the green line:



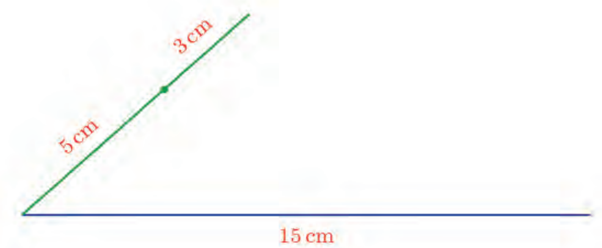
Another problem:

Draw a rectangle of perimeter 30 centimetres and sides in the ratio 5 : 3.

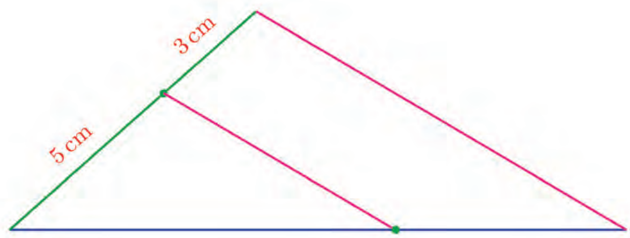
Since the perimeter is 30 centimetres, the sum of the lengths of the sides is 15 centimetres.

So, the length and breadth of the rectangle are the pieces got by dividing a 15 centimetre long line in the ratio 5 : 3.

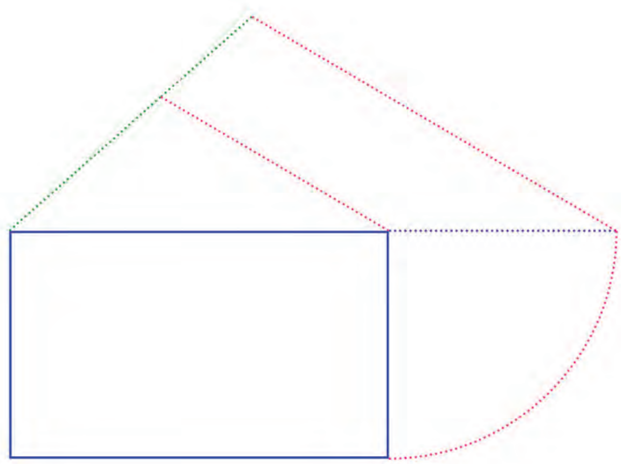
To get these pieces, we start like this:




Then join the other ends of these lines and draw the parallel line:



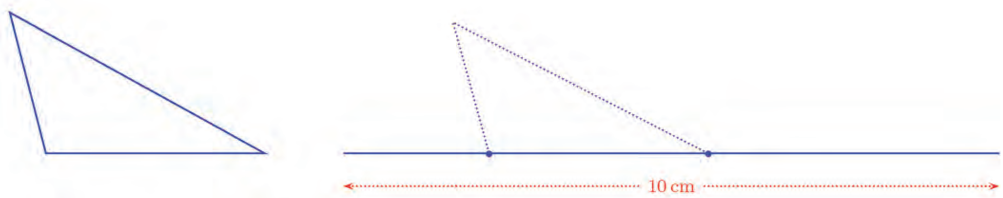

The rectangle we want is the one with the pieces of the bottom (blue) line as width and height:



One more problem:

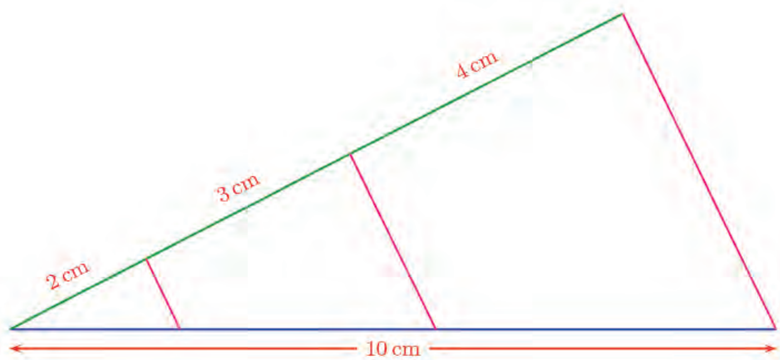
Draw a triangle of perimeter 10 centimetres and the ratio of the sides 2 : 3 : 4

If we unfold the sides of the triangle and spread them along a single straight line, it would be 10 centimetre long:

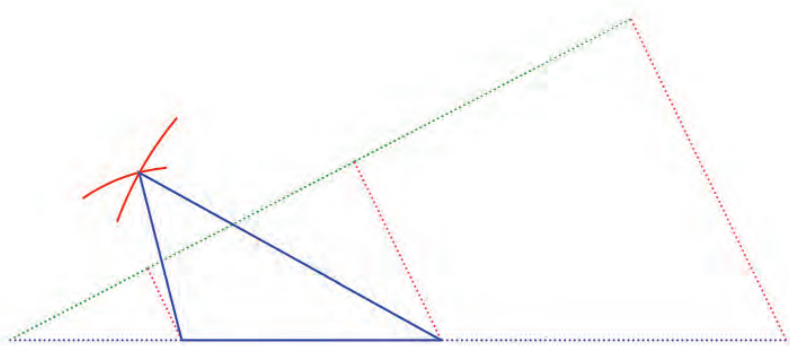



So, to draw the triangle, all we need to do is to first draw a line 10 centimetre long, then divide it in the ratio 2 : 3 : 4 and finally fold the pieces at the end.

What is the length which can be easily divided in this ratio?




Now we can draw the triangle:



(1) Draw a rectangle of perimeter 15 centimetres and sides in the ratio 3: 4.

(2) Draw a triangle of each of the types below, of perimeter 13 centimetres:

(i) Equilateral

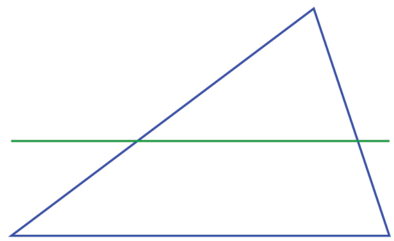
(ii) Sides in the ratio 3 : 4 : 5

(iii) Isosceles with lateral sides one and a half times the base

(3) Prove that in any trapezium, the diagonals cut each other in the same ratio.

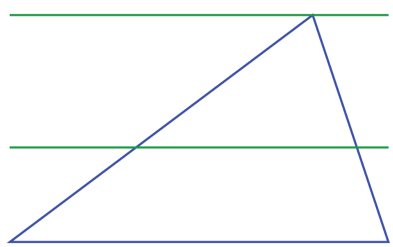
## Triangle division

Draw a triangle and a line within it, parallel to one side:



Is there any relation between the pieces into which this line cuts the other sides ?

Suppose we draw another line parallel to the bottom side through the top vertex:

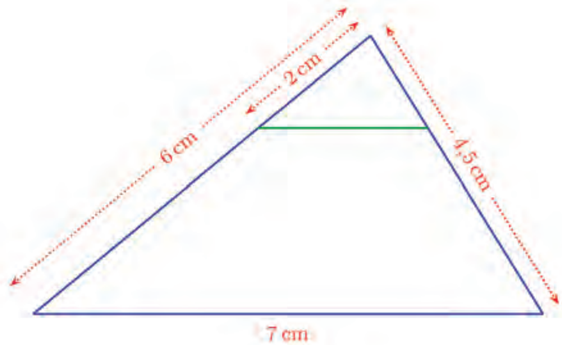


Now there are three parallel lines cutting the left and right sides of the triangle. The ratio of the pieces on either side must be the same, right?

What do we see here?

In any triangle, a line parallel to one of the sides cuts the other two sides in the same ratio.

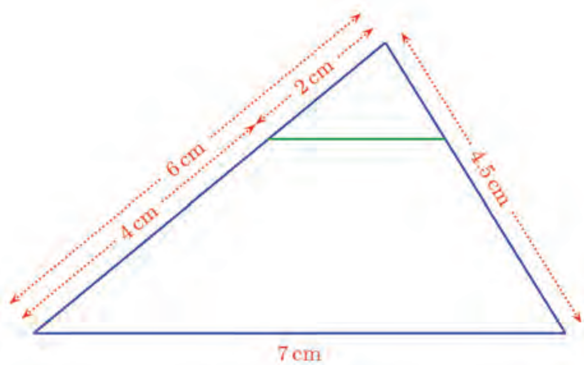
Let's look at a problem:




In this picture, the green line within the blue triangle is parallel to the bottom side. We want to calculate the ratio in which this line divides the right side.

It divides both left and right sides in the same ratio.

What's the ratio of the left pieces ?



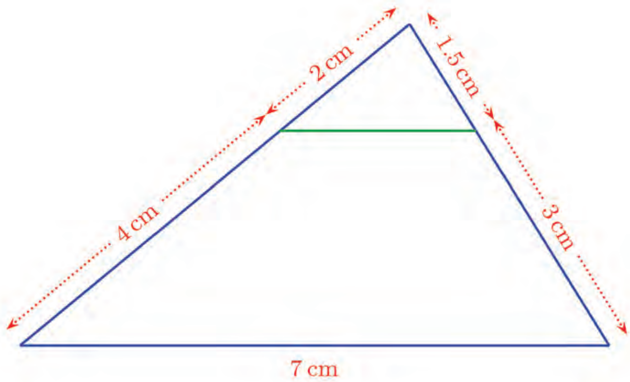

Construct a slider $a$ with Min = 0 and Max = 1 and Increment = 0.01. Draw a triangle $ABC$. Select **Enlarge from Point (Dilate from Point)** and click on $C$ and then on $A$. In the dialogue window, type the Scale Factor as $a$. We get a point $C'$ on the line $AC$. The length of $AC'$ would be $a$ times that of $AC$. (If $a = 0.5$, then $C'$ would be the midpoint of $AC$. If $a = 0.1$, then the lengths of $AC'$ and $CC'$ would be in the ratio 1 : 9. Mark the lengths of these lengths and check.) Draw a line parallel to $AB$ through $C'$ and mark the line where it meets $BC$ as $D$. Mark the lengths $BD$ and $DC$. Compare the ratios $AC' : C'C$ and $BD : DC$.

The longer piece on the left is twice the shorter piece. In other words, the ratio of the pieces is 1:2

So, the right side is also cut in this ratio.

Length of the shorter piece $= 4.5 \times \frac{1}{3} = 1.5$ cm

Length of the longer piece $= 4.5 \times \frac{2}{3} = 3$ cm




Did you notice another thing here?

In both the left and right sides, the shorter piece is $\frac{1}{3}$ of the whole side; and the longer piece is $\frac{2}{3}$ of the whole side.

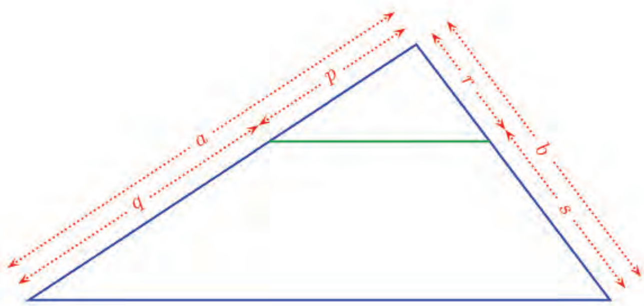
What if both sides are cut in the ratio 3: 5 ?

In both sides, the shorter piece would be $\frac{3}{8}$ of the side and the longer piece $\frac{5}{8}$ of the side.

So, the result stated earlier can also be put this way:

In any triangle, a line parallel to one side cuts the other two sides into the same parts.

See this picture:



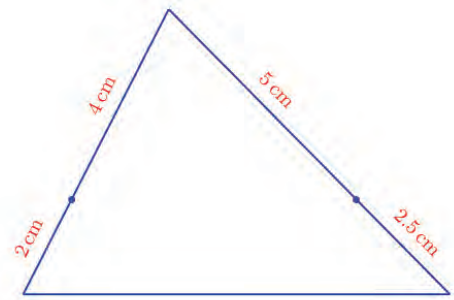

The green line within the triangle is parallel to the bottom side. Denoting the lengths of the sides and their parts by letters as in the picture, we can write the statement above as:

$$\frac{p}{a} = \frac{r}{b} \qquad \frac{q}{a} = \frac{s}{b}$$

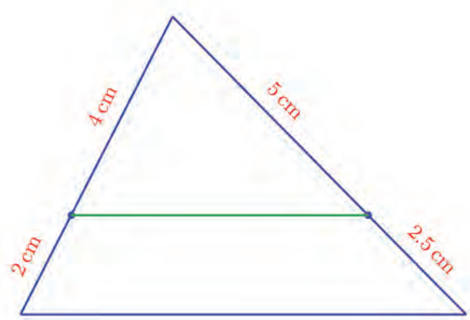
We've seen that a line parallel to one side of a triangle cuts the other two sides in the same ratio. Is the statement in reverse true?

That is, if a line divides two sides of a triangle in the same ratio, can we say that it is parallel to the third side?

For example, in the picture below, points dividing the left and right sides in the ratio 1 : 2 are marked:

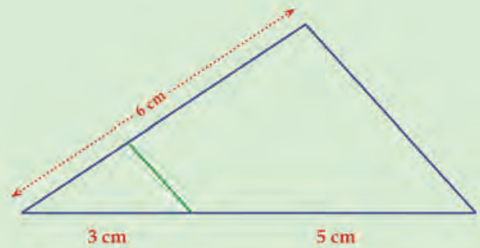



The line through the point on the left line, parallel to the bottom side, must divide the right side also in the ratio 1: 2. So, it must pass through the point marked on the right side. This means the line parallel to the bottom side, through the point on the left is the line joining these points. That is to say, the line joining these points is parallel to the bottom sides.



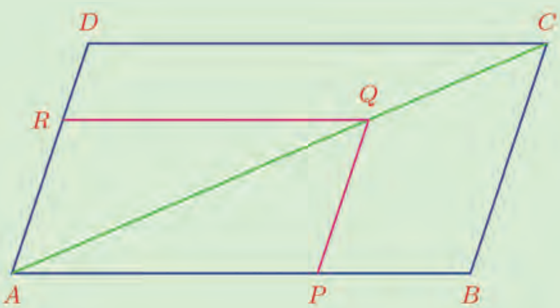

In any triangle, a line dividing two sides in the same ratio is parallel to the third side.

(1) In the picture below, the green line is parallel to the right side of the blue triangle.




Calculate the lengths of the pieces into which this line cuts the left side.

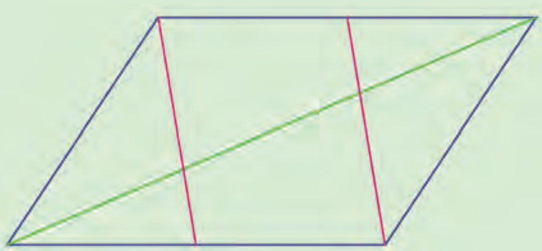
(2) In the parallelogram *ABCD*, the line through the point *P* on *AB*, parallel to *BC* meets *AC* at *Q*. The line through *Q* parallel to *AB* meets *AD* at *R*:




(i) Prove that $\frac{AP}{PB} = \frac{AR}{RD}$

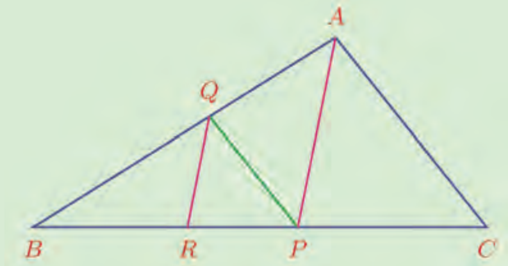
(ii) Prove that $\frac{AP}{AB} = \frac{AR}{AD}$

(3) In the picture below, two corners of a parallelogram are joined to the midpoints of two sides.



Prove that these lines cut the diagonal in the picture into three equal parts.

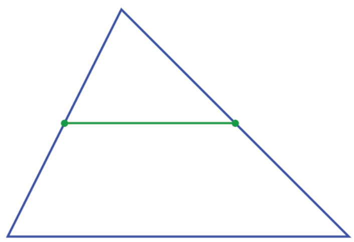
(4) In triangle *ABC*, the line parallel to *AC* through the point *P* on *BC*, meets *AB* at *Q*. The line parallel to *AP* through *Q* meets *BC* at *R*:




Prove that $\frac{BP}{PC} = \frac{BR}{RP}$

## Midsection

See this picture:



The green line within the triangle is the line parallel to the bottom side, through the midpoint of the left side.

Since it cuts the left side into equal pieces, it must also cut the right side into equal pieces. In other words, it meets the right side at its midpoint.

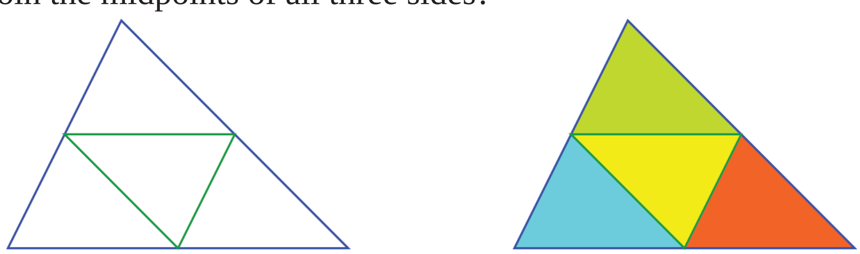
On the other hand, the line joining the midpoints of the left and right sides cuts both sides in the same ratio 1 : 1 and so is parallel to the bottom side.

This also is something which deserves special mention:

In any triangle, the line parallel to a side, through the midpoint of another side, meets the third side also at its midpoint.

On the other hand in any triangle, the line joining the midpoints of two sides is parallel to the third side

What if we join the midpoints of all three sides?



> Draw a triangle in GeoGebra and mark the midpoints of the sides. Draw the triangle connecting these midpoints. Mark the lengths of the sides of the large and small triangles. What is the relation between these lengths? Move the corners of the large triangle and check.

What can we say about the four small triangles? The sides of the yellow triangle in the middle are parallel to the sides of the large triangle containing all the small triangles.

Don't all these four triangles seem to be equal? Let's check this :

First look at the blue and yellow triangles. The right side of the blue one and the left side of the yellow triangle are the same.

The angle in the blue triangle at the top of this line, and the angle in the yellow triangle at the bottom of this line are equal (why?); so are the bottom angle of the blue triangle and the top angle of the yellow triangle.

So, these two triangles are equal. In the same way, we can see that the green and red triangles are also equal to the yellow triangle. Thus, all the four triangles are equal.

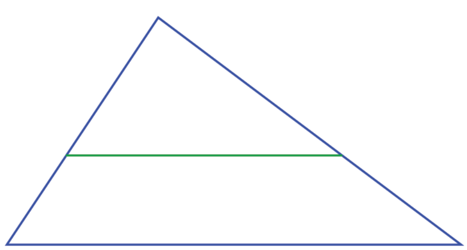
We can also see another thing from this. Since the sides of all these triangles are the same, each side is half the length of a side of the large triangle. Thus we have this result:

> In any triangle, the length of the line joining the midpoints of two sides is half the length of the third side.

We've also seen that the line joining the midpoints of two sides is parallel to the third side. This, together with the above result gives this:

> In any triangle, the line joining the midpoints of two sides is parallel to the third side and is half its the length.

We can look at this result in another manner. A line parallel to one side of a triangle makes a smaller triangle within the original one.



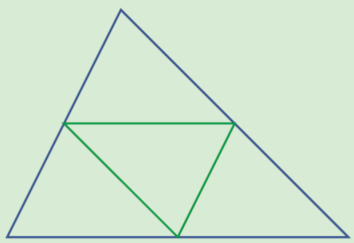
Two sides of this smaller triangle are on two sides of the large triangle; and they are the same parts of these sides.

What we've seen just now is that, if each of these sides is half of the side of the large triangle, then so is the third side.

What if the line cuts each of the two sides into a third? Would the remaining side of the smaller triangle also be a third of the remaining side of the large triangle?

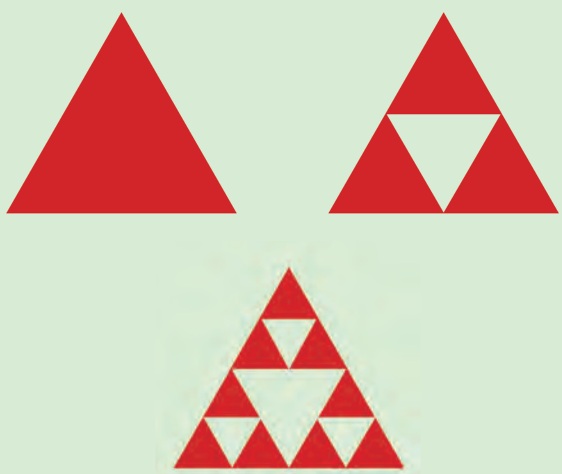
We will later see that whatever be the part of the two sides that the line cuts them, its length would be the same part of the third side.

(1) In the picture, the midpoints of a triangle are joined to form a smaller triangle inside:



How many times the perimeter of the small triangle is the perimeter of the large triangle? What about their areas?

(2) See these pictures:



The first picture is that of a triangle cut out from a sheet of paper. The second one shows it with the small triangle in the middle, joining the midpoints of the sides, cut off from the large triangle.

## Quarter problem

Each small triangle in the picture below is $\frac{1}{4}$ of the large triangle:

The three triangles at the bottom make $\frac{3}{4}$ of the large triangle and the yellow one at the top is the remaining $\frac{1}{4}$. What about this picture?

The blue, red and green together make $3\left(\frac{1}{4}+\frac{1}{4^2}\right)$, the yellow is $\frac{1}{16}$. And in this picture?

The blue, red and green triangles gradually fill up the whole triangle, right ? In arithmetical language this means the numbers,

$$\frac{3}{4}\ ;\ 3\left(\frac{1}{4}+\frac{1}{4^2}\right),\ 3\left(\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}\right)$$

and so on get closer and closer to 1. So, the numbers

$$\frac{1}{4},\ \frac{1}{4}+\frac{1}{4^2},\ \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}$$

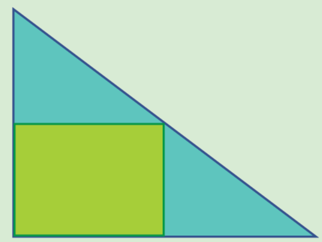
and so on get nearer and nearer to $\frac{1}{3}$.

The third picture shows such middle pieces cut off from each of the small triangles in the second picture

(i) What fraction of the area of the paper in the first picture is the area of the paper in the second picture?

(ii) What about in the third picture?

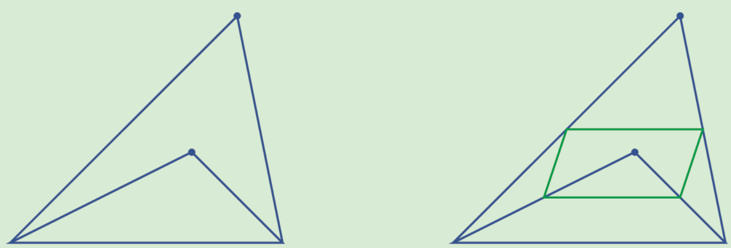
(3) A quadrilateral is drawn with the midpoints of the sides of a right triangle and its square corner as vertices.



(i) Prove that this quadrilateral is a rectangle.

(ii) What fraction of the area of the triangle is the area of the rectangle?

(4) In the two pictures below, the first one shows two triangles formed by joining the ends of a line to two points above it. The second one shows the quadrilateral formed by joining the midpoints of the left and right sides of these triangles:



(i) Prove that this quadrilateral is a parallelogram.

(ii) Describe the positions of the points on top for this quadrilateral to be:

(a) Rhombus (b) Rectangle

(c) Square

(iii) Would we get all these, if one point is taken above the line and one below?

(5) (i) Prove that the quadrilateral formed by joining the midpoints of any quadrilateral is a parallelogram.

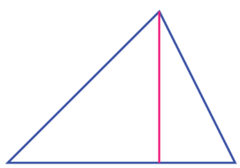
Draw a line $AB$ and mark two pints $C$ and $D$ above it. Draw two triangles by joining these points to $A$ and $B$. Mark the midpoints of the left and right sides of these triangles and draw the quadrilateral joining these points. What speciality of this quadrilateral do you notice ? Change the positions of $C$ and $D$ and see what happens. See at what positions of these points does the quadrilateral become a rhombus, rectangle or a square. Can we get these shapes if $C$ and $D$ are on different sides of $AB$ ?

(ii) Explain what should be the original quadrilateral to get the inner quadrilateral as

(a) Rhombus (b) Rectangle (c) Square

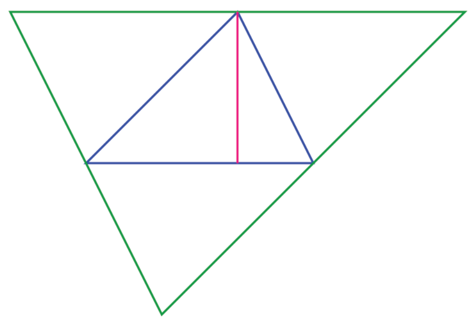
## Triangle centres

See this picture:



The perpendicular from the top vertex of the triangle to the opposite side is drawn. Such perpendiculars can be drawn from any vertex to the opposite side. These perpendiculars are called the altitudes (heights) of the triangle.

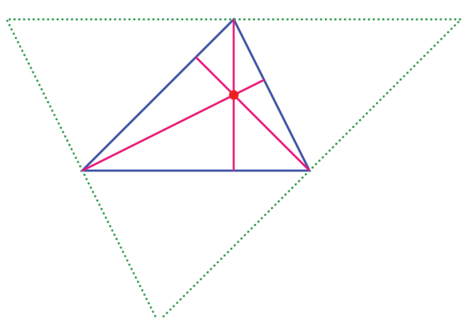
Now let's draw through each vertex, the line parallel to the opposite side:



Then the red line is not only perpendicular to the top side of the large triangle, it also bisects it.

So, if we draw the other altitudes of the smaller triangle, they would be the perpendicular bisectors of the other two sides of the larger triangle.

We have seen that all three perpendicular bisectors of the three sides of any triangle cross one another at a single point



Draw a triangle in GeoGebra and draw the perpendiculars from each vertex to the opposite side. Do they intersect at a single point ? Mark this point (orthocentre). Mark the angles of the triangle also. Now change the positions of the vertices. Is the orthocentre within the triangle in all cases ? What happens if one of the angles of the triangle is a right angle ? What if one angle is larger than a right angle ?

The perpendicular bisectors of the large triangle are the altitudes of the smaller triangle.

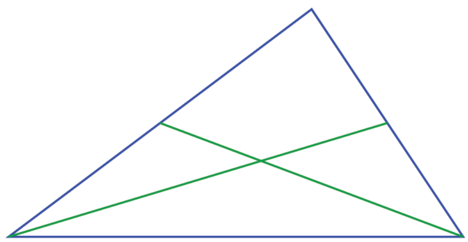
What do we see here ?

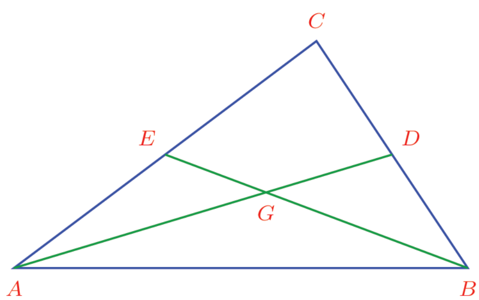
**In any triangle, all the altitudes intersect at a single point.**

The point where all the altitudes meet is called the *orthocentre* of the triangle.

Perpendicular bisectors of sides and altitudes are some of the special lines related to a triangle. Lines joining each vertex with the midpoint of the opposite side form another type of such lines. We have noted earlier that such lines are called medians of a triangle. Do they also intersect at a single point?

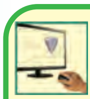
See this picture:






Two medians of a triangle are drawn. What we want to check is whether the third median also passes through the point of intersection of the two we have drawn.

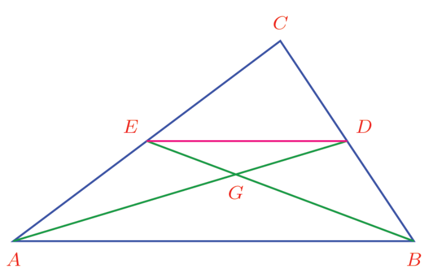
We can draw and check. But we need a proof that this is so in all triangles. To begin with, we name the points in the picture, for easy reference.



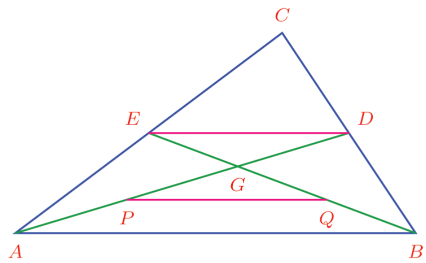
Draw a triangle in GeoGebra and mark the midpoints of the sides. Join each vertex to the midpoint of the side to get the medians. Do they all pass through a single point ? Mark this point (centroid). Mark the distance from each vertex to the centroid and also the distance from the centroid to the midpoint of the side. What is the relation between these distances? Change the positions of the vertices of triangle and check.

The line joining the midpoints of the left and right sides is parallel to the bottom side and is half its length. That is

$$ED = \frac{1}{2}AB$$



How about joining the midpoints of the small triangle GAB also?




We get

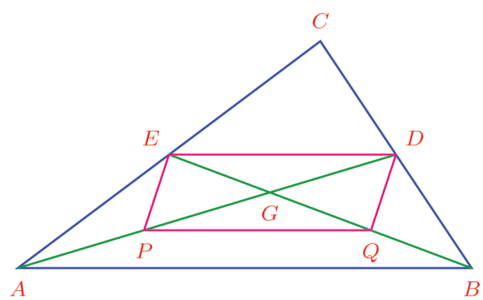
$$PQ = \frac{1}{2}AB$$

Thus we have

$$PQ = ED$$

Since the opposite sides *PQ* and *ED* of the quadrilateral *PQDE* are parallel and equal, it is a parallelogram. So, its diagonals *PD* and *QE* bisect each other. That is,

$$PG = GD \qquad QG = GE$$




Since *P* is the midpoint of *AG* and *Q* is the midpoint of *BG*, we have

$$AP = PG \qquad BQ = QG$$

So,

$$AG = AP + PG = 2PG = 2GD$$

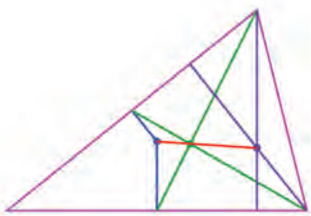
and

$$BG = BQ + QG = 2QG = 2GE$$

Draw a triangle in GeoGebra and mark the centroid (If we draw the triangle using the **Polygon** tool, then we can mark the centroid using the **Midpoint** or **Centre** tool). Draw a triangle joining the centroid and two vertices. We can draw three such triangles. Find the areas of these triangles. Is there any relation between them ? Change the positions of the vertices and check.

## Euler line

The famous mathematician Euler proved in the eighteenth century that in any triangle, the circumcentre, centroid and orthocentre lie along the same straight line. This line is now known as the Euler line.



In the picture above, the blue lines are the perpendicular bisectors of two sides of the triangle. Their point of intersection is the circumcentre. The green lines are medians and their point of intersection is the centroid. The violet lines are altitudes and their point of intersection is the orthocentre.

The red line passing through these three points is the Euler line.

Euler also proves that in any triangle, the centroid is between the circumcentre and orthocentre and the centroid divides the line joining them in the ratio 1: 2.

Draw a triangle in GeoGebra and find its circumcentre, centroid and orthocentre. Draw the line joining any two of them. Does it pass through the third point also ? Mark the lengths of the sides of the triangle. Change the positions of the vertices of the triangle. What happens to the Euler line when two sides of the triangle are equal? What happens to the three points when all three sides are equal ?

This shows that the point $G$ divides the lines $AD$ and $BE$ in the ratio 2 : 1.

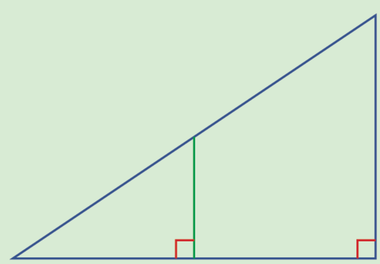
Now suppose instead of joining the vertices $A$ and $B$ to the midpoints of their opposite sides, we join the vertices $B$ and $C$ to the midpoints of their opposite sides. Their points of intersection would divide $BE$ in the ratio 2 : 1. In other words, their point of intersection is also $G$.

So we get this result:

In any triangle, all three medians intersect at a single point. The distance of this point from each vertex is twice the distance from the midpoint of the opposite side. In other words, this point divides each median in the ratio 2 : 1.

This point of intersection of the medians is called the centroid of the triangle.

(1) Draw a right triangle and draw the perpendicular from the midpoint of the hypotenuse to the base:



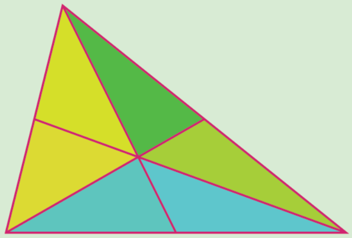
(i) Prove that this perpendicular is half the vertical side of the large triangle.

(ii) Prove that the distances from the midpoint of the hypotenuse to the three vertices of the large triangle are equal.

(iii) Prove that the circumcentre of a right triangle is the midpoint of its hypotenuse.

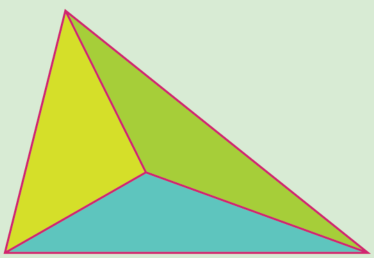
(2) Prove that in any equilateral triangle, the circumcentre, orthocentre and the centroid coincide.

(3) In the picture below, the medians of the triangle divide it into six small triangles:



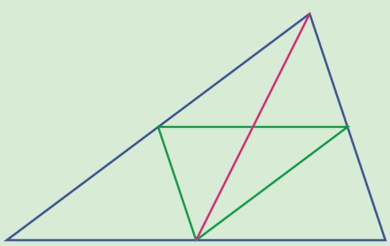
Prove that all six triangles have the same area.

(4) In the picture below, the triangle is divided into three small triangles by joining the centroid to the three vertices:



Prove that all three triangles have the same area.

(5) In the picture below, the midpoints of the sides of the blue triangle are joined to make the smaller green triangle. The red line is a median of the large triangle.



## Balancing act

In physics, we often have to assume that the entire mass of a body is concentrated at a single point. It is called the centre of mass. It is physically experienced as the point at which the body can be balanced.

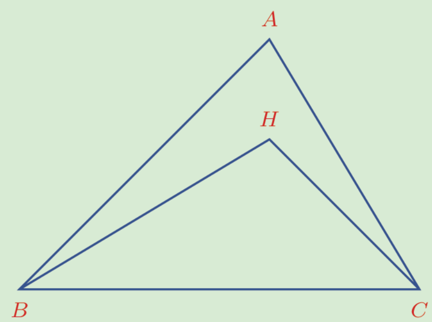
The centre of mass of a triangle cut out from a thick sheet of paper or a thin sheet of metal is its centroid.

Draw a triangle in GeoGebra and mark the midpoints of the sides. Draw the altitudes from each vertex and mark the points where they meet the opposite sides. Find the orthocentre of the triangle. Mark the midpoints of the lines joining the orthocentre and each vertex. Draw the circle through these three points (The **Circle through 3 Points** tool can be used for this). Aren't the other six points we marked, the midpoints of the sides and the feet of the altitudes, also on this circle ? This circle which passes through nine points of the triangle is called its Nine-point circle.

(i) Prove that this median bisects the top side of the small triangle.

(ii) Prove that the centroid of the large and small triangles are the same.

(6) In the picture below, *H* is the orthocentre of triangle *ABC*.




Prove that *A* is the orthocentre of triangle *HBC*