5 Irrational Multiplication

We have seen that the length of the diagonal of a square cannot be expressed as a fractional multiple of the length of its side. In other words, if we take the side of a square as the unit of length, then its diagonal cannot be expressed as a fraction.

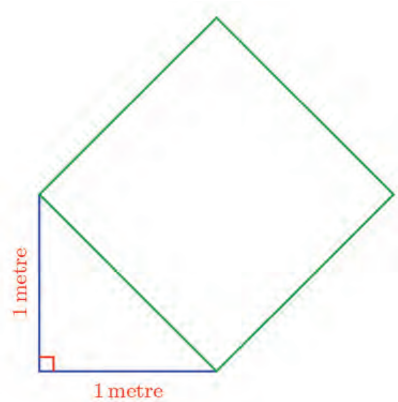
We have also seen that this length is denoted by the symbol $\sqrt{2}$. We then saw some other lengths like these and new numbers to denote them. (We will see another such number in the lesson, **Circular Measures**).

Such numbers which are used to denote lenghts which cannot be expressed as fractions of a unit are called *irrational numbers*.

We have also seen how joining lengths denoted by irrational numbers leads to the definition of addition of such numbers. We now see how computation of areas lead to the operation of multiplication of such numbers.

Multiplication

We've seen this picture so many times:



What is the perimeter of the square in this?

We know that the length of each of its sides is $\sqrt{2}$ metres. So, its perimeter is four times this length.

That is, $\sqrt{2}+\sqrt{2}+\sqrt{2}+\sqrt{2}$

As in the case of other numbers, we can write 4 times $\sqrt{2}$ as $4\times\sqrt{2}$ or $\sqrt{2}\times 4$. Usually this is written without the multiplication symbol, as $4\sqrt{2}$. Thus

$$4\sqrt{2} = \sqrt{2}+\sqrt{2}+\sqrt{2}+\sqrt{2}$$

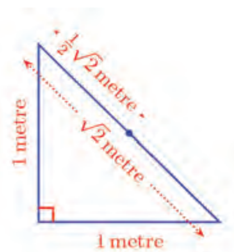
To find fractions approximating this number, we multiply by 4, fractions approximating $\sqrt{2}$:

$$4 \times 1.4, \quad 4 \times 1.41, \quad 4 \times 1.414, \ldots$$

So the perimeter of our square up to a millimetre is

$$4 \times 1.414 = 5.656 \text{ metres}$$

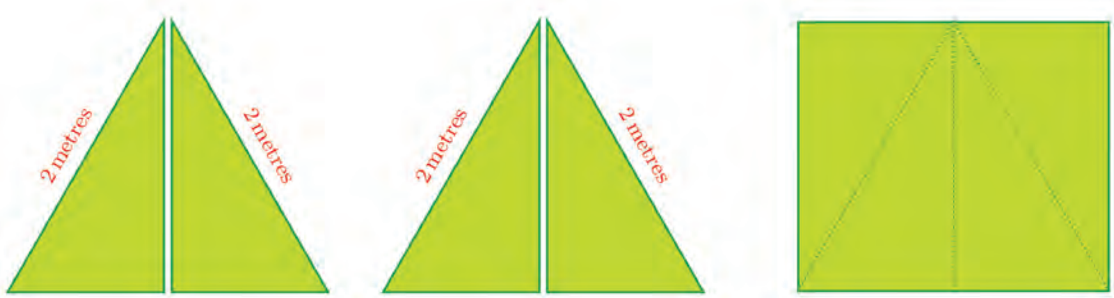
In the same way half of $\sqrt{2}$ is written as $\frac{1}{2}\sqrt{2}$.



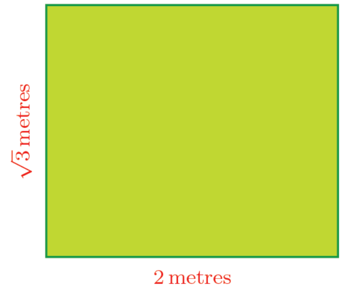
As before, to get fractions approximating $\frac{1}{2}\sqrt{2}$, we take half of fractions approximating $\sqrt{2}$. That is

$$\frac{1}{2}\sqrt{2} = 0.7071\ldots$$

Now see these pictures:



Two equilateral triangles of the same size are cut into right triangles and the pieces are rearranged to form a rectangle.



What is the area of this rectangle?

We know the length of its sides.

We've seen that if the lengths of the sides of a rectangle are fractions, then its area is the product of those numbers.

Is the area in this case also the product of its sides ? That is, $2\sqrt{3}$ square metres.

To check this, let's draw rectangle within this, all with one side 2 metres and the other sides of lengths approximating $\sqrt{3}$ metres:

As we continue with inner rectangles of heights 1.73 metres, 1.732 metres and so on, we find their areas as twice these lengths.

Geometrically these rectangles gradually fill up the outer rectangle. So, their areas get nearer and nearer to the area of the outer rectangle.

In other words, the numbers

$$2 \times 1.7, \qquad 2 \times 1.73, \qquad 2 \times 1.732, \ldots$$

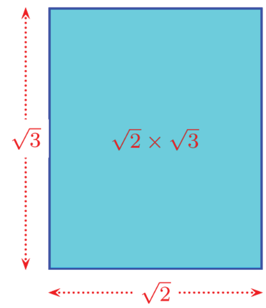
get closer and closer to the area of the outer rectangle.

According to our definition, these numbers get closer and closer to $2\sqrt{3}$.

Thus we see that the area of a rectangle of sides 2 and $\sqrt{3}$ is $2\sqrt{3}$.

Now what about a rectangle of sides $\sqrt{2}$ and $\sqrt{3}$?

We denote this area by the symbol $\sqrt{2} \times \sqrt{3}$.



To describe this as a number, we multiply the fractions approximating $\sqrt{2}$ and $\sqrt{3}$ in order and take up to the required number of decimal places:

$$\begin{array}{llll} 1.4 & 1.41 & 1.414 & \ldots \rightarrow \sqrt{2} \\ 1.7 & 1.73 & 1.732 & \ldots \rightarrow \sqrt{3} \\ \hline 2.4 & 2.44 & 2.449 & \ldots \rightarrow \sqrt{2} \times \sqrt{3} \end{array}$$

That is,

$$\sqrt{2} \times \sqrt{3} = 2.449\ldots$$

A question naturally arises here.

We know that the product of square roots, which are natural numbers or fractions, is equal to the square root of the product.

For example, we can compute $\sqrt{4} \times \sqrt{25}$ in two ways: either as product of the roots

$$\sqrt{4} \times \sqrt{25} = 2 \times 5 = 10$$

or as root of the product

$$\sqrt{4} \times \sqrt{25} = \sqrt{4 \times 25} = \sqrt{100} = 10$$

In the same manner

$$\sqrt{\frac{1}{4}} \times \sqrt{\frac{9}{25}} = \frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$$

$$\sqrt{\frac{1}{4}} \times \sqrt{\frac{9}{25}} = \sqrt{\frac{1}{4} \times \frac{9}{25}} = \sqrt{\frac{9}{100}} = \frac{3}{10}$$

Decimal math

In approximating up to a decimal place, if the digit after the place we want to cut is larger than or equal to 5, then we add 1 more to the digit at which we cut. For example, since $1.4 \times 1.7 = 2.38$, we take this product up to one decimal place as 2.4.

Like this, is $\sqrt{2} \times \sqrt{3}$ equal to $\sqrt{6}$?

To check this, we must check if the squares of the numbers, 2.4, 2.44, 2.449 and so on, which approximate $\sqrt{2} \times \sqrt{3}$, get closer and closer to 6:

$$2.4^2 = 5.76$$

$$2.44^2 \approx 5.95$$

$$2.449^2 \approx 5.998$$

Since these squares get closer and closer to 6, we have

$$\sqrt{6} = 2.449\ldots$$

and since we also have

$$\sqrt{2}\times\sqrt{3} = 2.449\ldots$$

we can conclude

$$\sqrt{2}\times\sqrt{3} = \sqrt{6}$$

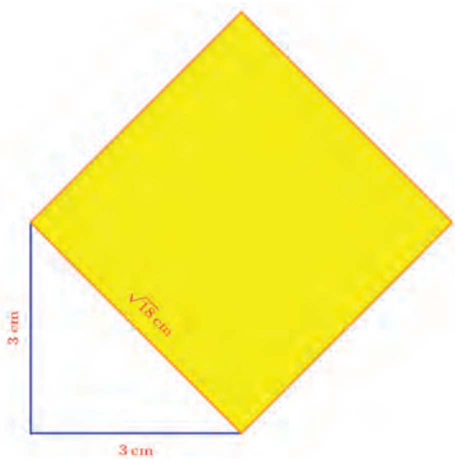
In the same way, we can see that for numbers other than 2 and 3 also, the product of the square roots is the square root of the product. Thus we have this result:

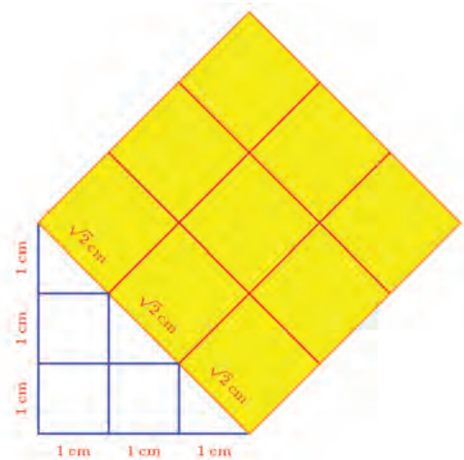
$\sqrt{x}\times\sqrt{y}=\sqrt{xy}$ for all positive numbers x and y

This can be used to simplify certain square roots. As an example, let's look at the hypotenuse of a right triangle with both the perpendicular sides 3 centimetres. By Pythagoras Theorem, the area of the square on the hypotenuse is $3^2 + 3^2 = 18$ square centimetres, so that the length of the hypotenuse is $\sqrt{18}$ centimetres. Now if we write 18 as 9 × 2, this becomes

$$\sqrt{18}=\sqrt{9\times2}=\sqrt{9}\times\sqrt{2}=3\sqrt{2}$$

We can also see this geometrically:





Product and square root

We have seen that the fractions 2.4, 2.44, 2.449 and so on approximate $\sqrt{2}\times\sqrt{3}$ and that the squares of these get closer and closer to 6. Why does this happen?

We got these fractions as approximations to the products

$$1.4 \times 1.7,\ 1.41 \times 1.73,\ 1.414 \times 1.732,\ \ldots$$

and so on. The squares of these are

$$(1.4 \times 1.7)^2,\ (1.41 \times 1.73)^2,\ (1.414 \times 1.732)^2,\ \ldots$$

and so on. We can write these as products of squares:

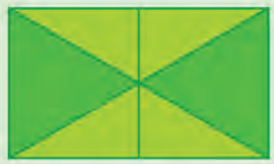
$$(1.4 \times 1.7)^2 = 1.4^2 \times 1.7^2$$

$$(1.41 \times 1.73)^2 = 1.41^2 \times 1.73^2$$

$$(1.414 \times 1.732)^2 = 1.414^2 \times 1.732^2$$

In the products on the right side of these products, the numbers 1.4^2, 1.41^2, 1.414^2 and so on get closer and closer to 2 and the numbers 1.7^2, 1.73^2, 1.732^2 and so on get closer and closer to 3. So, their products get closer and closer to 6.

(1) We can make a rectangle using four equilateral triangles of the same size by cutting two of them along their heights and rearranging these pieces and the other two whole triangles:

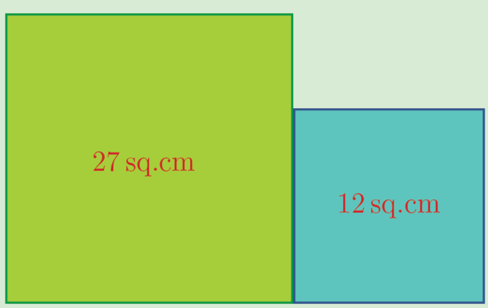


If the sides of all equilateral triangles are 2 centimetres, what is the perimeter and area of the rectangle ?

(2) We can make a trapezium by cutting a square and an equilateral triangle with sides twice that of the square, and rearranging the pieces as below:

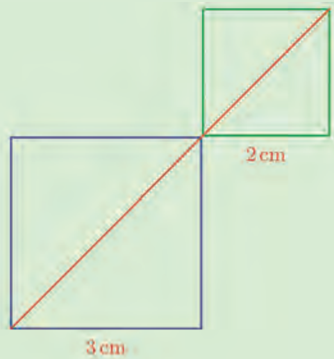
If the side of the square is 2 centimeters, what is the perimeter and area of the trapezium?

(3) The picture shows the figure formed by joining two squares:



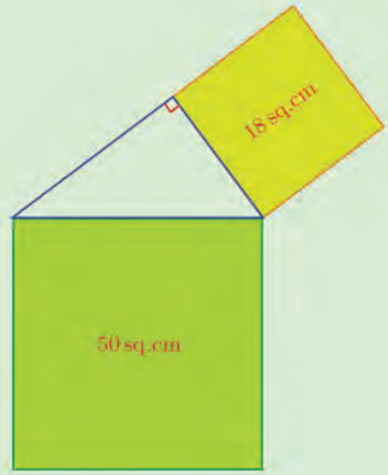
Calculate the length of the bottom side of this figure, correct to a centimetre.

(4) The figure shows two squares with two corners joined:



Find the length of the slanted line.

(5) Calculate the length of the third side of the right triangle in the picture and also its perimeter.



(6) The product of some of the pairs of numbers below are natural numbers or fractions. Find those pairs.

(i) $\sqrt{3}, \sqrt{12}$ (ii) $\sqrt{3}, \sqrt{1.2}$ (iii) $\sqrt{5}, \sqrt{8}$

(iv) $\sqrt{0.5}, \sqrt{8}$ (v) $\sqrt{7\frac{1}{2}}, \sqrt{3\frac{1}{3}}$ (vi) $\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{10}}$

Division

From the product $2 \times 3 = 6$, we get two quotients, $\frac{6}{2} = 3$ and $\frac{6}{3} = 2$

Similarly, we can write the product $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ also as two quotients

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \qquad \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

In general for any x and y which are natural numbers or fractions, the product $x \times y = z$ gives two quotients $\frac{z}{x} = y$ and $\frac{z}{y} = x$

Like this,

For any two positive numbers x and y, the product

$$\sqrt{x} \times \sqrt{y} = \sqrt{z}$$

can be written as the quotients

$$\frac{\sqrt{z}}{\sqrt{x}} = \sqrt{y} \qquad \frac{\sqrt{z}}{\sqrt{y}} = \sqrt{x}$$

Now since $\frac{6}{2} = 3$ and $\frac{6}{3} = 2$ we have

$$\sqrt{\frac{6}{2}} = \sqrt{3}\,, \qquad \sqrt{\frac{6}{3}} = \sqrt{2}$$

and what did we see earlier ?

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \qquad \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

From these two equations, we see that

$$\sqrt{\frac{6}{2}} = \frac{\sqrt{6}}{\sqrt{2}} \qquad \sqrt{\frac{6}{3}} = \frac{\sqrt{6}}{\sqrt{3}}$$

Similarly, we can write

$$\sqrt{3} \times \sqrt{\frac{2}{3}} = \sqrt{3 \times \frac{2}{3}} = \sqrt{2}$$

and from this, the quotient

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

For any two positive numbers x and y

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

Next let's see how such square roots are computed. For example, to compute $\sqrt{\frac{1}{2}}$, we first write

$$\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Then divide 1 by some decimal approximation of $\sqrt{2}$ to get an approximation of $\sqrt{\frac{1}{2}}$.

$$\frac{1}{\sqrt{2}} \approx \frac{1}{1.414} = 0.707$$

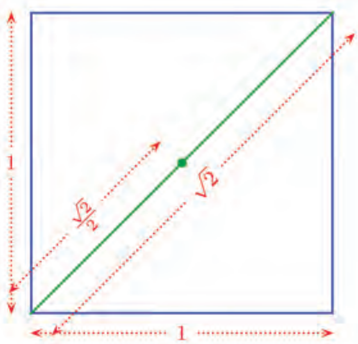
There's a simpler way. Since $\frac{1}{2} = \frac{2}{4}$, we can do the computation like this:

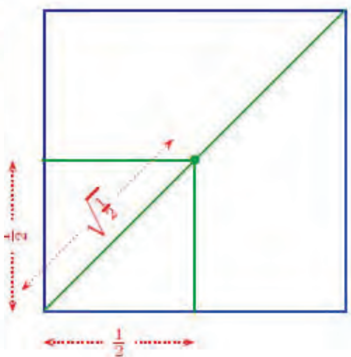
$$\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

Now we can easily calculate

$$\frac{\sqrt{2}}{2} \approx \frac{1.414}{2} = 0.707$$

We can also see geometrically that $\frac{\sqrt{2}}{2} = \sqrt{\frac{1}{2}}$:





Similarly we can compute

$$\frac{1}{\sqrt{3}} = \sqrt{\frac{1}{3}} = \sqrt{\frac{3}{9}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{3}}{3}$$

In general,

For any positive number x

$$\frac{1}{\sqrt{x}} = \sqrt{\frac{1}{x}} = \sqrt{\frac{x}{x^2}} = \frac{\sqrt{x}}{\sqrt{x^2}} = \frac{\sqrt{x}}{x}$$

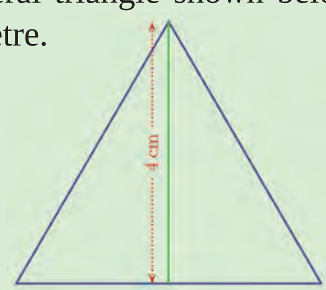
(1) Prove that $(\sqrt{2}+1)(\sqrt{2}-1)=1$.

Using this:

(i) Compute $\frac{1}{\sqrt{2}-1}$ up to two decimal places.

(ii) Compute $\frac{1}{\sqrt{2}+1}$ up to two decimal places.

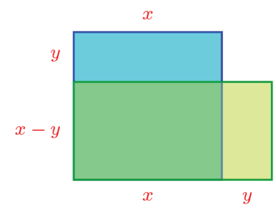
(2) Compute the lengths of the sides of the equilateral triangle shown below, correct to a millimetre.



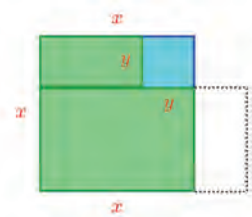
Multiplication identity

Remember the geometry of the identity $(x+y)(x-y) = x^2 - y^2$?

Consider a square with lengths of side x. Suppose that we increase one side by y and decrease another side by y, to form a rectangle:



The area of this rectangle is $(x + y)(x - y)$. We now cut the part of the rectangle jutting out on the right and place it on top:



The area of the green part of the picture is $x^2 - y^2$.

This is equal to the area of the green rectangle we drew first, isn't it ? Area of the rectangle is the product of sides even if the lengths are irrational. So, the identity $(x + y)(x - y) = x^2 - y^2$ is true for irrational numbers also. Similarly we can show that other identities are also true for irrational numbers.

(3) All red triangles in the picture are equilateral and of the same size. What is the ratio of the sides of the outer and inner squares?

(4) Prove that $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$ and $\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}$. Find other numbers like this.

(5) Among the pairs of numbers given below, find those for which the quotient of the first by the second is a natural number or a fraction.

(i) $\sqrt{72}, \sqrt{2}$ (ii) $\sqrt{27}, \sqrt{3}$ (iii) $\sqrt{125}, \sqrt{50}$

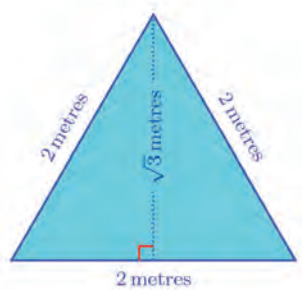
(iv) $\sqrt{10}, \sqrt{2}$ (v) $\sqrt{20}, \sqrt{5}$ (vi) $\sqrt{18}, \sqrt{8}$

Areas of triangles

We defined numbers like $\sqrt{2}$ and $\sqrt{3}$ to denote certain lengths which cannot be expressed as fractions. Later we saw that such numbers are needed to denote the areas of some rectangles also.

If the lengths of the sides of a rectangle are natural numbers or fractions, so would be the area. But the area of triangles with the lengths of all sides natural numbers or fractions may not be such a number.

For example, we have seen that the height of an equilateral triangle with all sides 2 metres is $\sqrt{3}$ metres.

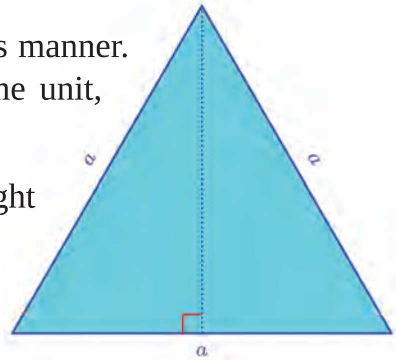


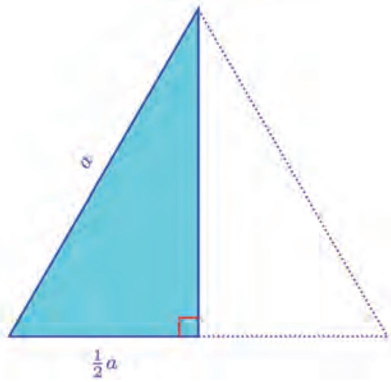
What is its area?

$\frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$ square metres

We can calculate the area of any equilateral triangle in this manner. If we take the length of the sides as a (measured in some unit, centimetre, metre or something else), what is its height?

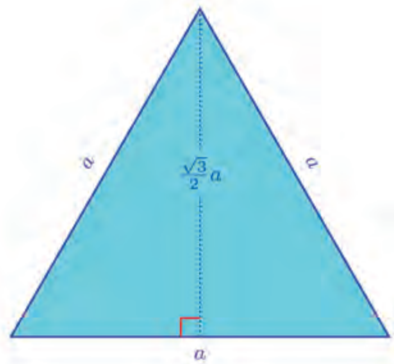
We can write down the hypotenuse and base of the two right triangles within it.





So, the square of the third side is

$$a^2 - \left(\frac{1}{2}a\right)^2 = \frac{3}{4}a^2$$



and from this, the length of side is

$$\sqrt{\frac{3}{4}a^2} = \frac{\sqrt{3}}{2}a$$

This is the height of the equilateral triangle.

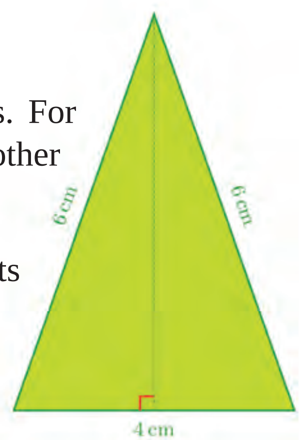
Now we can calculate the area:

$$\text{Area} = \frac{1}{2} \times a \times \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{4}a^2$$

Thus we have this result:

> The altitude of an equilateral triangle is $\sqrt{3}$ times half the side and its area is $\sqrt{3}$ times the square of half the side.

For example, the height of an equilateral triangle with all sides 8 centimetres is $4\sqrt{3}$ centimetres and its area is $16\sqrt{3}$ square centimetres.



We can also calculate the areas of isosceles triangles like this. For example, take the triangle with one side 4 centimetres and the other two 6 centimetres each.

In this triangle also, the perpendicular from the top vertex bisects the base, right ? (The lesson **Equal Triangles** in Class 8)

So, we can compute the square of the height, as we did for equilateral triangles:

$$\text{Square of height} = 6^2 - 2^2 = 32$$

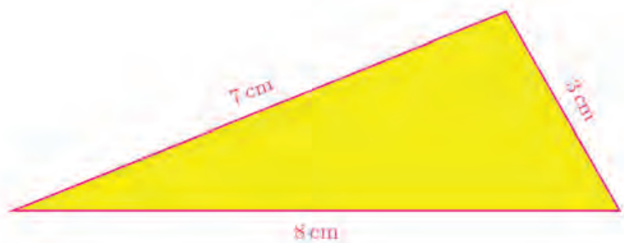
So the height is

$$\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2} \text{ cm}$$

Now we can calculate the area:

$$\text{Area} = \frac{1}{2} \times 4 \times 4\sqrt{2} = 8\sqrt{2} \text{ sq.cm}$$

In much the same way, we can calculate the area of any triangle (requires more computation, that's all). For example, look at this triangle:



First we have to calculate its height. Let's take it as h centimetres and the length of one of the pieces it cuts the base as x centimetres:

Then from the right triangle on the left, we get

$$x^2 + h^2 = 49$$

and from the right triangle on the right,

$$(8 - x)^2 + h^2 = 9$$

Subtracting the second equation from the first, we get

$$x^2 - (8 - x)^2 = 40$$

We have seen in Class 8 that we can write the difference of two squares as the product of sum and difference. Using this,

$$x^2 - (8 - x)^2 = (x + (8 - x))\,(x - (8 - x))$$

$$= 8(2x - 8)$$

So our equation becomes

$$8(2x - 8) = 40$$

From this, we get $2x - 8 = 5$ and then

$$x = \frac{1}{2}(5 + 8) = 6\frac{1}{2}$$

Now again from the left triangle, we get

$$h^2 = 7^2 - \left(6\frac{1}{2}\right)^2 = 13\frac{1}{2} \times \frac{1}{2}$$

$$= 6\frac{1}{2} + \frac{1}{4} = 6\frac{3}{4}$$

so that

$$h = \sqrt{6\frac{3}{4}} = \sqrt{\frac{27}{4}} = \frac{3}{2}\sqrt{3}$$

Using this, we can calculate the area

$$\text{Area} = \frac{1}{2} \times 8 \times \frac{3}{2}\sqrt{3} = 6\sqrt{3} \text{ sq.cm}$$

In these computations, if we take the lengths of the sides as a, b, c we get a formula to compute the area directly in terms of a, b, c without involving the height:

> If the lengths of the sides of a triangle are a, b, c and we take
>
> $$s = \frac{1}{2}(a + b + c)$$
>
> then the area of the triangle is
>
> $$\sqrt{s(s-a)(s-b)(s-c)}$$

(If you are interested in knowing how this formula is got, see the appendix at the end of this chapter)

This formula was discovered by Heron, a Greek mathematician of the sixth century.

As an example, in the problem we did just now, if we take

$$a = 8 \qquad b = 7 \qquad c = 3$$

then

$$s = \frac{1}{2}(8 + 7 + 3) = 9$$

We next calculate

$$s - a = 1$$
$$s - b = 2$$
$$s - c = 6$$

and put them in Heron's formula to get

$$\text{Area} = \sqrt{9 \times 1 \times 2 \times 6} = \sqrt{9 \times 4 \times 3} = 6\sqrt{3}$$

(1) For each of the lengths below, calculate the area of the equilateral triangle with that as the lengths of the sides:

(i) 10 cm (ii) 5 cm (iii) $\sqrt{3}$ cm

(2) Calculate the area of the regular hexagon with lengths of the sides 6 centimetres.

(3) Calculate the perimeter and area of the equilateral triangle with height 12 centimetres.

(4) Calculate the perimeter and area of the regular hexagon with the distance between parallel sides 6 centimetres.

(5) Calculate the height and area of the triangle with sides 8 centimetres, 6 centimetres, 6 centimetres.

(6) For each of the set of three lengths given below, calculate the area of the triangle with these as the lengths of sides:

(i) 4 cm, 5 cm, 7cm

(ii) 4cm, 13 cm, 15 cm

(iii) 5 cm, 12 cm, 13 cm

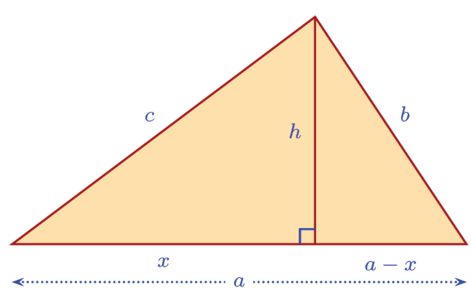
Land area

One method of calculating the area of a piece of land, with all boundaries straight, is to divide it into triangles and measure the various lengths:

The the area of each triangle can be computed using Heron's formula and their sum gives the area of the piece of land.

Appendix

Let's see how we get Heron's formula for the area of a triangle. Let us denote the lengths of the sides of the triangle as a, b and c. As we did in an earlier problem, we take the length of the perpendicular from one vertex to the opposite side as h and the length of one of the pieces in which this perpendicular cuts the side as x.



Then from the right triangle on the left we get.

$$x^2 + h^2 = c^2$$

and from that on the right,

$$(a - x)^2 + h^2 = b^2$$

Subtracting the second equation from the first, we get

$$x^2 - (a - x)^2 = c^2 - b^2$$

Next we write the difference of squares $x^2 - (a - x)^2$ as the product of sum and difference:

$$x^2 - (a - x)^2 = (x + (a - x))\,(x - (a - x)) = a(2x - a)$$

So the equation got from the triangle can be written as :

$$a(2x - a) = c^2 - b^2$$

From this we get

$$2x - a = \frac{c^2 - b^2}{a}$$

which gives

$$2x = \frac{c^2 - b^2}{a} + a = \frac{c^2 - b^2 + a^2}{a}$$

Thus we get

$$x = \frac{c^2 - b^2 + a^2}{2a}$$

Now in the equation

$$h^2 = c^2 - x^2 = (c + x)\,(c - x)$$

got from the left triangle, we write x in terms of a, b, c given by the previous equation:

$$h^2 = \left(c + \frac{c^2 - b^2 + a^2}{2a}\right)\left(c - \frac{c^2 - b^2 + a^2}{2a}\right)$$

This can be simplified as below:

$$h^2 = \left(\frac{2ac + (c^2 - b^2 + a^2)}{2a}\right)\left(\frac{2ac - (c^2 - b^2 + a^2)}{2a}\right)$$

The numerator of the first fraction on the right side of the equation can be simplified like this:

$$\begin{aligned}
2ac + (c^2 - b^2 + a^2) &= (a^2 + c^2 + 2ac) - b^2 \\
&= (a + c)^2 - b^2 \\
&= ((a + c) + b)\,((a + c) - b) \\
&= (a + c + b)\,(a + c - b)
\end{aligned}$$

And the numerator of the second fraction like this:

$$\begin{aligned} 2ac - (c^2 - b^2 + a^2) &= 2ac - c^2 + b^2 - a^2 \\ &= b^2 - (a^2 + c^2 - 2ac) \\ &= b^2 - (a - c)^2 \\ &= (b + a - c)(b - (a - c)) \\ &= (b + a - c)(b - a + c) \end{aligned}$$

Now we can write h^2 using these:

$$h^2 = \frac{(a+c+b)(a+c-b)(b+a-c)(b-a+c)}{4a^2}$$

Now if we write p for the perimeter of the triangle, then

$$p = a + b + c$$

so that

$$\begin{aligned} p - 2a &= (a + b + c) - 2a = (b + c - a) \\ p - 2b &= (a + b + c) - 2b = (a - b + c) \\ p - 2c &= (a + b + c) - 2c = (a + b - c) \end{aligned}$$

So we can write

$$h^2 = \frac{p(p-2b)(p-2c)(p-2a)}{4a^2}$$

Next if we denote half the perimeter (semiperimeter) as s, then $p = 2s$, so that we haves

$$\begin{aligned} h^2 &= \frac{2s(2s-2b)(2s-2c)(2s-2a)}{4a^2} \\ &= \frac{2s \times 2(s-b) \times 2(s-c) \times 2(s-a)}{4a^2} \\ &= \frac{4s(s-b)(s-c)(s-a)}{a^2} \end{aligned}$$

From this we get

$$h = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}$$

Finally we can calculate the area:

$$\begin{aligned} \text{Area} &= \frac{1}{2}ah = \frac{1}{2} \times a \times \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a} \\ &= \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$