6

SIMILAR TRIANGLES

Angles and sides

We know that if the sides of a triangle are equal to the sides of another triangle, then their angles are also equal. We have also noted that on other hand, even if angles of two triangles are all equal, the sides may not be equal (The lesson, **Equal Triangles** in Class 8). This raise the question: Is there any relation between the sides of two triangles with the same angles ?

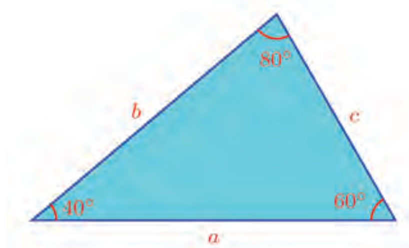
To check this, cut out two triangles with the same angles but different sides from a thick sheet of paper. For examples, two triangles like these :

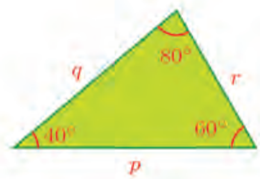
To compare the sides, place the smaller triangle over the larger, with the top corners aligned. Since the angles are equal, the sides making these corners will also be aligned:

Now the bottom sides of both the triangles are equally inclined to the left side. So these sides are parallel.

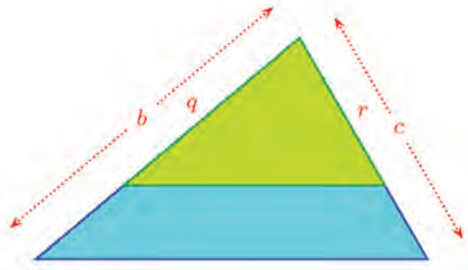
So, the bottom side of the smaller triangle divides the left and right sides of the larger triangle into the same parts of each (The lesson, **Parallel Lines**).

To state this concisely, we denote the lengths of the sides of the triangle by letters as below:





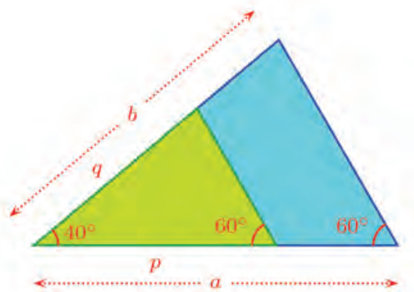
When the smaller triangle is placed over the larger as before, we can mark the lengths as below:



Then the fact (about same parts on either side) found earlier can be written like this:

$$\frac{q}{b} = \frac{r}{c}$$

Suppose we align the left corners instead of the top ones?



In the same manner as above, we can now see that the right sides of the triangles are now parallel, and so the right side of the smaller triangle divides the left and bottom sides of the larger triangle into the same parts of each. That is

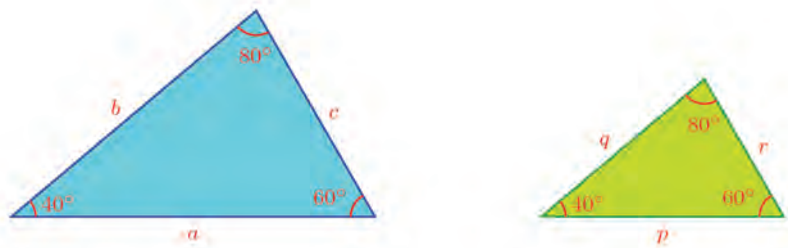
$$\frac{p}{a} = \frac{q}{b}$$

This together with the equation got earlier give

$$\frac{p}{a} = \frac{q}{b} = \frac{r}{c}$$

What does this equation tell us?

First let's look again at what exactly the letters represent:



- a and p are the lengths of the sides opposite the 80° angle
- b and q are the lengths of the sides opposite the 60° angle
- c and r are the lengths of the sides opposite the 40° angle

Next let's see what the fractions in the equation mean:

- The number $\frac{p}{a}$ gives what part of the length a is the length p
- The number $\frac{q}{b}$ gives what part of the length b is the length q
- The number $\frac{r}{c}$ gives what part of the length c is the length r

So what the equation

$$\frac{p}{a} = \frac{q}{b} = \frac{r}{c}$$

means is that all these parts are the same.

That is, if we pair the sides opposite the equal angles of the two triangles as (p, a), (q, b) and (r, c), then the shorter lengths p, q, r are the same part of the longer lengths a, b, c.

We can also say instead that the longer lengths are same times the smaller lengths. This would be true, whatever be the three angles instead of 80°, 60°, 40°.

Thus we have the general result:

In two triangles with the same angles, if we pair the sides opposite equal angles, then the shorter lengths are all the same part of the longer one (or the longer lengths are all same times the shorter).

In any triangle, the shortest side is opposite the smallest angle and the longest side is opposite the largest angle. In other words, the lenghts of the sides follow the same order as the size of the angles.

Draw the triangle *ABC* in GeoGebra and mark all its angles. Create a slider *d* with Min = 0. Use the **Segment with Given Length** tool to draw a line of length *d* times that of *AB*. For this, give the length of the line as *d*AB*. Next draw triangle *DEF* with $\angle D = \angle A$ and $\angle E = \angle B$. For this, select the **Angle with Given Size** tool and click on the points *E* and *D* in order and in the dialogue window give he measure of the angle as α (the measure of $\angle A$). Similarly, click on *D* and *E* in order and give the measure of the angle as ß, with the option clockwise selected. Join the lines *DE*' and *ED*' and mark the point of intersection *F*. Mark the lengths of the sides of both triangles. Are they in the same ratio ? Change the lengths of the sides of triangle *ABC* and the value of the slider and check

Using this idea, we can shorten the statement of the above result like this:

In triangles with the same angles, the sides, in the order of lengths, are in the same ratio.

We can state this in another manner also. We use the term scaling, in representing one measure as a multiple of another. For example take a 6 centimetre long line and a 4 centimetre long line. The longer length is $1\frac{1}{2}$ times the other, while the shorter one is $\frac{2}{3}$ of the other. We say that the longer line is got by scaling (up) the shorter by a factor of $1\frac{1}{2}$ and the shorter is got by scaling (down) the longer by a factor of $\frac{2}{3}$. We can also say that the scale factor from the shorter to the longer is $1\frac{1}{2}$, while the scale factor from the longer to the shorter is $\frac{2}{3}$.

We've seen that if the triangle with lengths of sides *a, b, c* and *p, q, r* have the same angles, then *p, q, r* are the same times *a, b, c*. This means the change from the sides of the first triangle to the second has the same scale factor. If this scale factor is taken as *k*, then the relation between the lengths of the sides can be written:

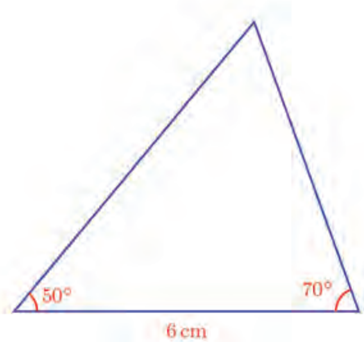
$$a = kp,\ b = kq,\ c = kr$$

So, the general result can be stated like this:

In triangles with the same angles, the sides opposite equal angles are scaled by the same factor.

Now look at this problem:

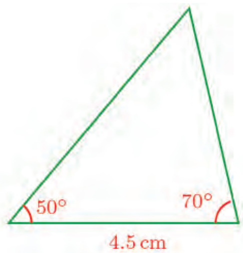
We want to draw a smaller triangle with all its sides $\frac{3}{4}$ of the sides of this triangle:



The bottom side of the triangle we want should be 4.5 centimetres long. What about the other sides?

Should we first draw the large triangle, measure its other two sides, and draw the smaller triangle with sides $\frac{3}{4}$ of this?

We need only draw a line 4.5 centimetre long and draw the same angles as those of the larger triangle, right ?



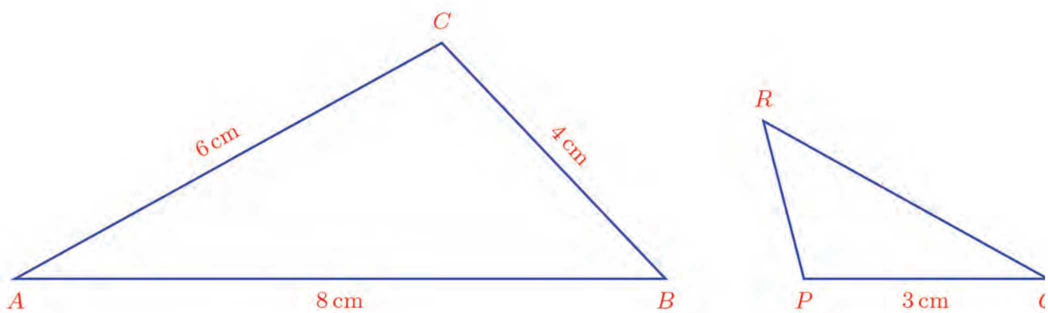
Since the angles are equal, the other sides will also be $\frac{3}{4}$ part of those of the larger triangle, by the general result we've seen.

Triangle speciality

If two triangles have the same angles, their sides are in the same ratio. This is a speciality not shared by other polygons. For example, look at these pictures:

In both the rectangle and the square, all angles are right. The lengths of the left sides of both are in the same (ratio 1 : 1); so are the right sides. But the top sides of the rectangle and the square are of different lengths; the bottom sides are also of different lengths.

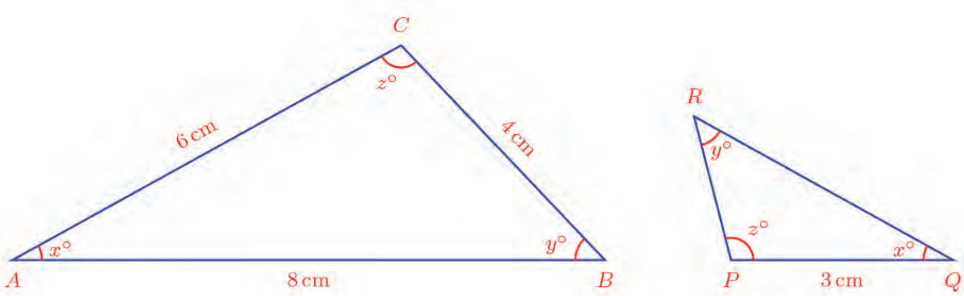
Look at another problem:



$$\angle P = \angle C \qquad \angle Q = \angle A \qquad \angle R = \angle B$$

How do we compute the lengths of the other two sides of the triangle PQR ?

First let's mark the equal angles, taking their measures as $x°$, $y°$, $z°$.



Next we write the pairs of sides opposite equal angles:

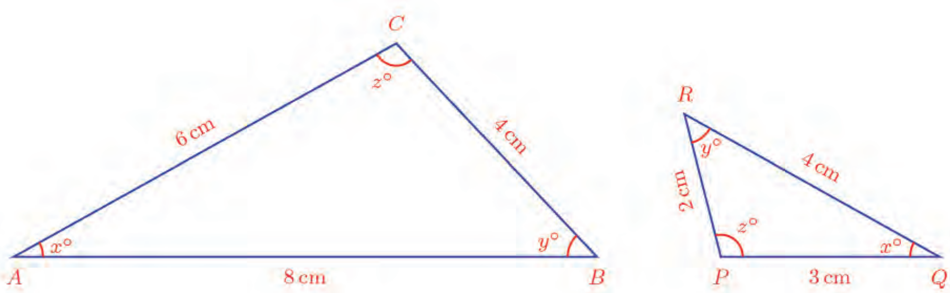
x	BC	PR
y	AC	PQ
z	AB	QR

We know the lengths of all sides of the larger triangle and the length of one side of the smaller:

x	$BC = 4$	PR
y	$AC = 6$	$PQ = 3$
z	$AB = 8$	QR

We see that for the sides opposite the $y°$ angle, the shorter side is half of the longer. So the sides opposite other angles must be scaled by the same factor:

x	$BC = 4$	$PR = 2$
y	$AC = 6$	$PQ = 3$
z	$AB = 8$	$QR = 4$



If we can compare the lengths of the sides of the smaller triangle just by sight, we can compute the lengths without writing the angles:

	Shortest	Medium	Longest
ΔABC	$BC = 4$	$AC = 6$	$AB = 8$
ΔPQR	PR	$PQ = 3$	QR

From this we can see that the sides of the smaller triangle are half those of the larger and using this, calculate the lengths of the other two sides.

(1) One side of a triangle is 8 centimetres and the two angles on it are 60° and 70°. Draw the triangle with lengths of sides $1\frac{1}{2}$ times that of this triangle and with the same angles.

(2) In a right triangle, the perpendicular from the square corner to the hypotenuse divides it into pieces 2 centimetres and 3 centimetres long:

(i) Prove that the two small right triangles formed by this perpendicular have the same angles

(ii) Taking the height of the perpendicular as h, prove that $\frac{h}{2} = \frac{3}{h}$.

(iii) Calculate the lengths of the perpendicular sides of the original large triangle.

(iv) Prove that if the length of the perpendicular from the square corner of a right triangle to the hypotenuse is h and it divides the hypotenuse into pieces of length a and b, then $h^2 = ab$.

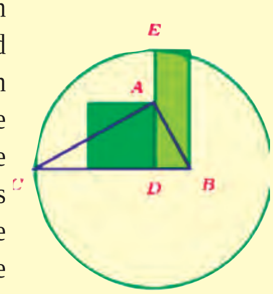
(3) At the two ends of a horizontal line, angles of the same size are drawn and two points on these slanted lines are joined :

(i) Prove that the horizontal line (blue) and the slanted line (red) cut each other into parts in the same ratio.

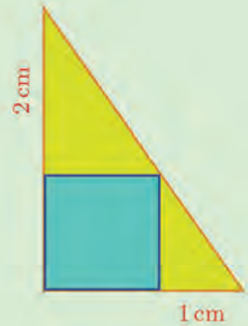
(ii) Prove that the slanted lines (green) at the ends of the horizontal line are also in the same ratio.

(iii) Explain how this idea can be used to divide a 6 centimetre long line in the ratio 3 : 4.

Draw right triangle ABC in GeoGebra and draw the perpendicular from the square corner A to the hypotenuse BC. Mark the point D where it meets the hypotenuse. Draw the circle with centre at D and passing through C and mark the point E where the perpendicular meets the circle. Draw the square with one side AD and the rectangle with sides BD and DE. Mark their areas. Aren't they equal ? Change the positions of the right triangle and check.



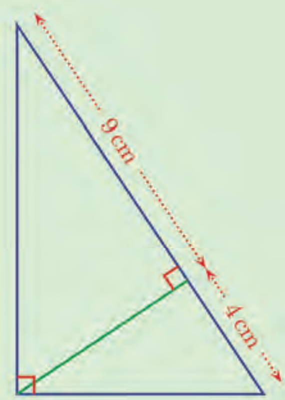
(4) The picture below shows a square sharing one corner with a right triangle and the other three corners on the sides of this triangle:



(i) Calculate the length of the sides of the square.

(ii) What is the length of the sides of such a square drawn within a triangle of sides 3 centimetres 4 centimetres, 5 centimetres?

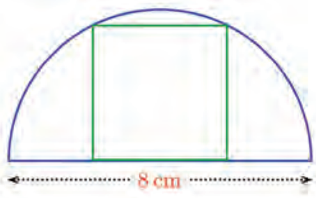
(5) Calculate the area of the largest right triangle in the picture below:



(6) Two poles of heights 3 metres and 2 metres are erected upright on the ground and ropes are stretched from the top of each to the foot of the other:

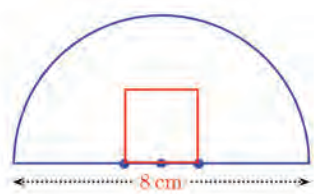
Semicircle and square

See this picture:

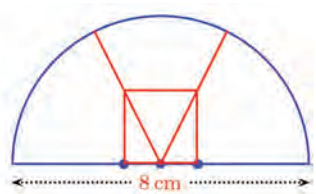


How do we inscribe a square within a semicircle like this?

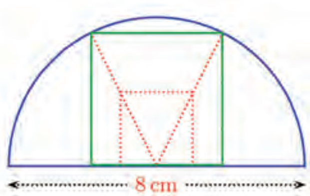
First mark two points on the diameter of the semicircle, equidistant from the centre. Draw a square on this part of the diameter:



Join the centre of the circle to the top corners of this square and extend these lines to meet the semicircle.



Join the points where these lines meet the semicircle. Draw perpendiculars to the diameter from these points:



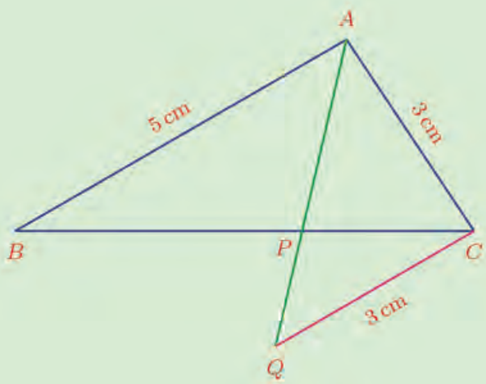
Can you prove that what we get thus is infact a square?

(i) At what height above the ground do the ropes cross each other ?

(ii) Prove that this height would be the same whatever the distance between the poles.

(iii) Denoting the heights of the poles as a, b and the height of the point of crossing above the ground as h, find the relation between a, b and h.

(7) In the picture below, AP is the bisector of $\angle A$ of triangle ABC:



(i) Prove that angles of the triangles ABP and CPQ are the same.

(ii) Calculate $\frac{BP}{PC}$

(iii) Prove that in any triangle, the bisector of an angle divides the opposite side in the ratio of the sides containing the angle.

Draw a horizontal line and mark points C and D on it. Draw perpendiculars to the first line through these. Mark a point E on the perpendicular through C and a point F on the perpendicular through D. Draw the lines ED and FC and mark their point of intersection G. Now change the positions of C and D and check the height of G above the horizontal line. Right click on G and select the Trace On. What is the path of G as the distance between C and D changes?

Draw triangle ABC in GeoGebra and draw the bisector of $\angle C$. Mark the point D where it meets AB. Compare the ratio of the lengths of the sides AC and BC and the ratio of the lengths AD and BD. Typing AC/BC in the **Input Bar** gives the number $\frac{AC}{BC}$. The number $\frac{AD}{BD}$ found like this.

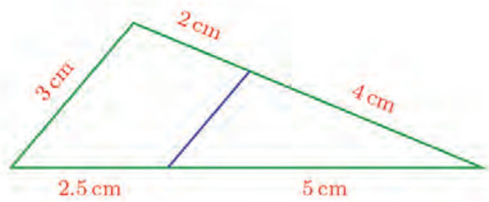
Sides and angles

We saw that if the angles of two triangles are the same, then their sides are scaled by the same factor. This raises the reverse question: if all sides of a triangle are scaled (stretched or shrunk) by the same factor, would the angles remain the same ?

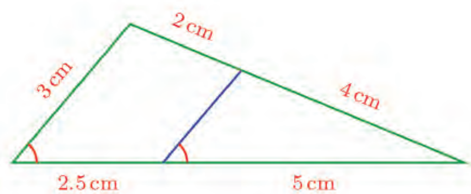
See this picture:

The sides of the larger triangle are all one and a half times those of the smaller triangle. Are the angles of the triangles the same ?

To check this, we first mark the lengths of two sides of the small triangle on the large triangle and join these points as shown below:



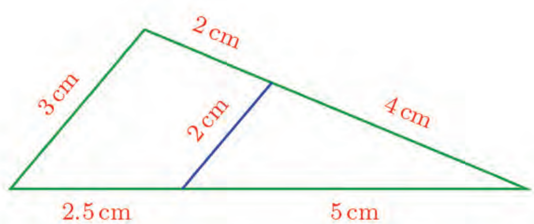
This line divides the bottom side and the right side of the large triangle in the same ratio 1 : 2. So, it is parallel to the left side (The section **Triangle division** in the lesson, **Parallel Lines**). This means their inclinations with the bottom side are the same.



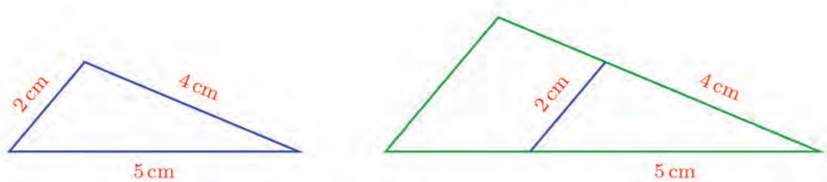
So, if just look at the large triangle and the small triangle within it (let's ignore the small triangle outside for the time being), we see that they have the same angles.

By the general result seen earlier, the sides of these two triangles are scaled by the same factor.

The bottom side of the small triangle is $\frac{2}{3}$ of the bottom side of the large triangle. The right sides are also scaled by the same factor. So, the left sides of the triangles must also be scaled by this factor. Thus we can calculate the third side of the small triangle.



Now let's look again at the small triangle outside the large triangle, which we had kept aside:



The small triangles within and out have sides of the same length and so their angles are also equal (The lesson, **Equal Triangles** in Class 8).

And we have seen that the angles of the large triangle are the same as those of the small triangle within it.

So what do we get?

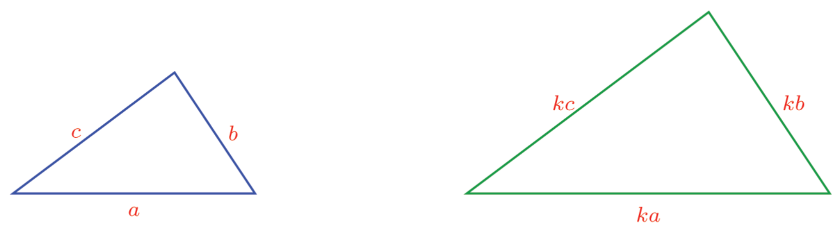
The angles of the small and large triangles we started with are the same.

Even if we change the lengths of the sides and the scale factor used in this example, we can show that the angles of the two triangles are the same, using the same arguments as above.

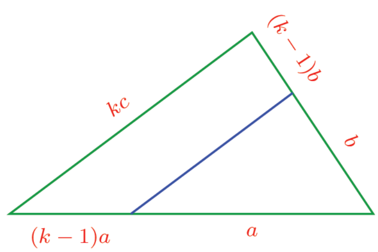
For those who need greater precision, we can do the same thing in a general setting using algebra.

Take two triangles with lengths of sides of one scaled by the same factor in the other. That is, the lengths of the sides of one triangle are multiples of those of the other by the same number.

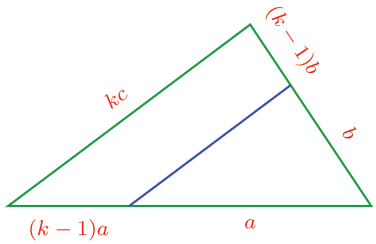
So, if we take the lengths of the sides of the small triangle as a, b, c, then those of the large triangle can be taken ka, kb, kc.



As we did in the example, we mark the lengths of two sides of the small triangle on two sides of the large triangle and join these points, as shown below:

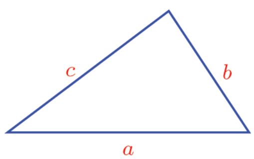


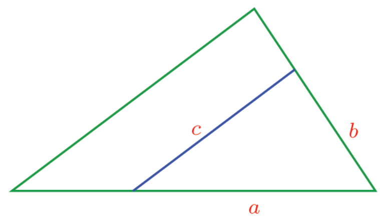
This line divides the bottom and right sides of the large triangle in the same ratio $(k - 1) : 1$. So, this line is parallel to the left side. So, the large triangle and the small one inside have the same angles.



This means their sides must be scaled by the same factor. The bottom side of the small triangle within is $\frac{1}{k}$ part that of the bottom side of the large triangle (same is the case with right sides). So, the left sides also must be scaled by the same factor:

Now as in the example, we look at the small triangles within and outside the large triangle:

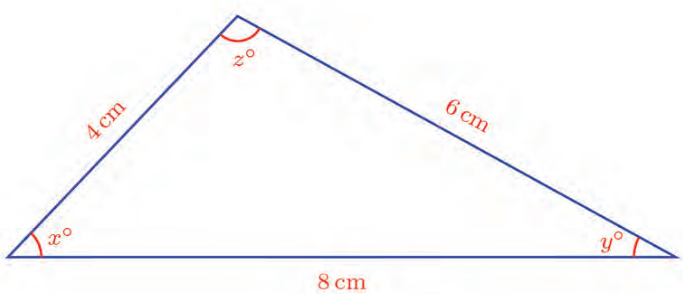


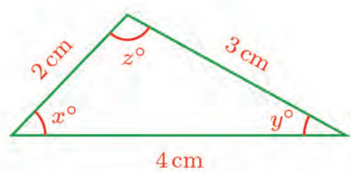


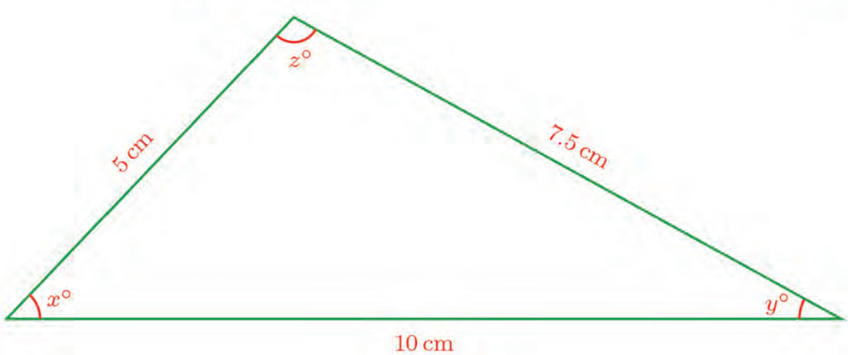
Since the lengths of the sides of these triangles are the same, the angles must also be the same; and we have seen that the angles of the large triangle are the same as those of the small triangle within. Thus the small and large triangles we started with have the same angles.

If two triangles have their sides scaled by the same factor, then their angles are the same.

So to enlarge or shrink a triangle without altering the angles, we need not measure the angles. We need only scale the sides by the same factor:



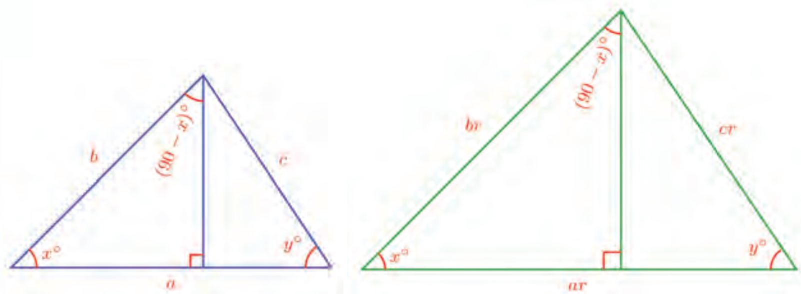




Let's look at a problem using this idea. We can easily see that if all sides of a triangle are scaled by the same factor, then the perimeter also is scaled by the same factor. (Try it!)

What about the areas?

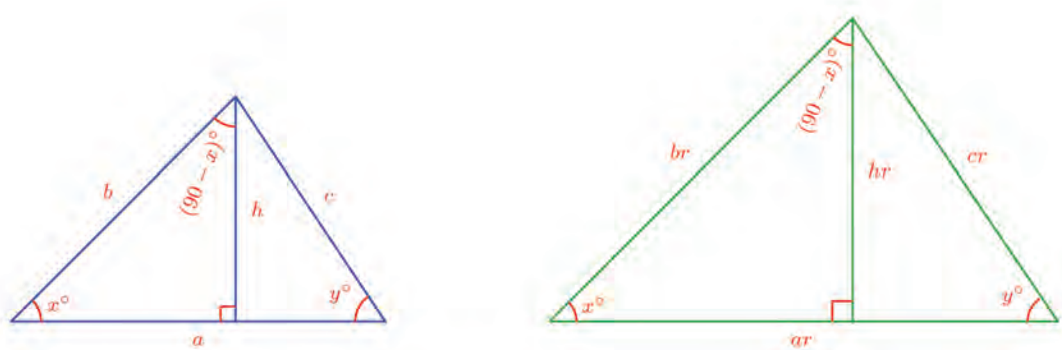
To see it, let's draw two such triangles. As we have seen just now, they have the same angles. To compare the areas, let's draw perpendiculars from two vertices with the same angles:



Draw triangle *ABC* in GeoGebra with names of the sides *a*, *b*, *c*. Create a slider *k* with Min=0. Draw line *DE* with length ka. With *D* and *E* as centres, draw circles of radii *ka* and *kb* and mark one of their points of intersection as *F*. Draw triangle *DEF*. Mark the angles of both triangles. Aren't they equal? Change the value of the slider and positions of the vertices of the first triangle. What do you see?

Mark the perimeter and area of the two triangles. What is the scale factor of the perimeters? And of the areas?

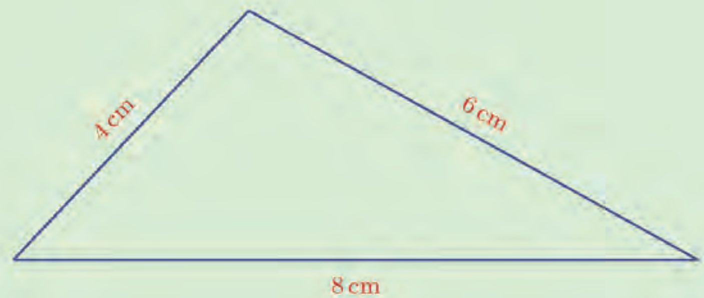
Focus on the two right triangles on the left. Both have the same angles, $x°$, $90°$, $(90-x)°$. So, their sides are scaled by the same factor. The hypotenuse of the blue triangle is b and that of the green is br. So, if the length of the vertical side of the blue triangle is taken as h, then the length of the vertical side of the green triangle is hr.



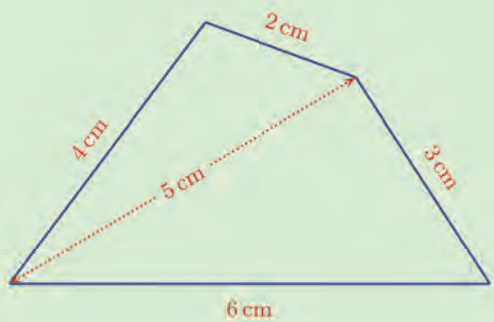
Now we can compute the areas of the whole triangles. The area of the blue triangle is $\frac{1}{2}ah$; that of the green triangle is $\frac{1}{2}ahr^2$.

Thus the scale factor of area is the square of the scale factor of the sides.

(1) Draw the triangle with angles the same as those of the triangle shown below, and sides scaled by $1\frac{1}{4}$.



(2) See the picture of the quadrilateral.



(i) Draw the quadrilateral with the same angles as this and sides scaled by a factor of $1\frac{1}{2}$.

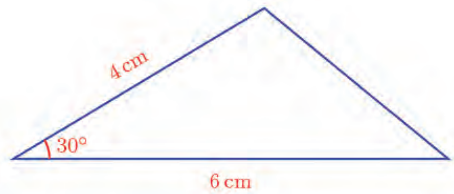
(ii) Draw a quadrilateral with angles different from this and sides scaled by a factor of $1\frac{1}{2}$.

(3) The area of a triangle is 6 square centimetres. What is the area of the triangle with lengths of sides four times those of this? What about the one with lengths of sides half of this?

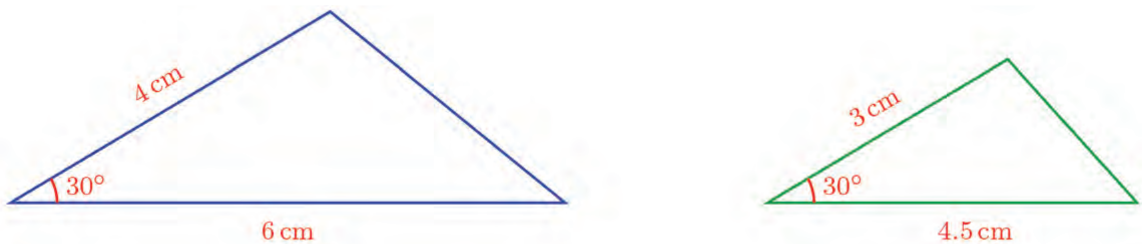
Third way

We saw in the first section how we can scale a triangle up or down without altering its angles, if we know the length of one side and two angles on it. Scale the known side by the required factor and draw the same angles at each end; the other sides would be scaled by the same factor. In the second section we saw how this could be done, if the lengths of the three sides are known instead: just scale all sides by the required factor; the angles would be the same.

Now suppose we want to do this on a triangle with lengths of two sides and the included angle known. For example see this picture. We want to scale it down by a factor of $\frac{3}{4}$ without altering the angles.

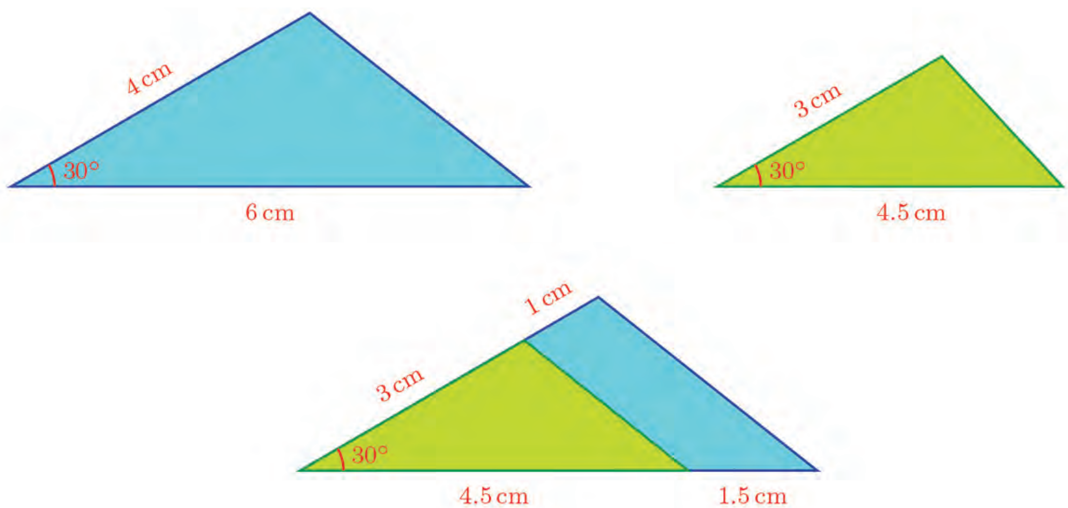


We can draw a triangle with two sides $\frac{3}{4}^{\text{th}}$ of 6 centimetres and 4 centimetres joined at an angle of 30°:

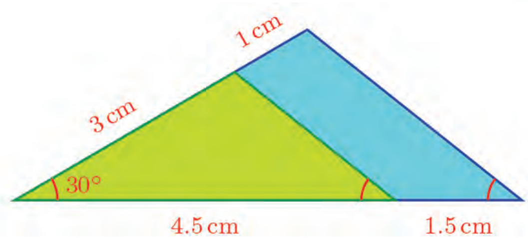


But we don't know whether the third side is also scaled down by $\frac{3}{4}$ and whether the other two angles of the triangles are the same.

To check this, cut out these triangles from a thick sheet of paper and place the smaller over the larger with the left corners aligned, as we did in the first section. Since the angles are equal, the sides meeting at this corner would also be aligned:



Now the right side of the green triangle divides the left and bottom sides of the blue triangle in the same ratio. So, it is parallel to the right side of the blue triangle. This means the right sides of the triangles are equally inclined to the bottom side.

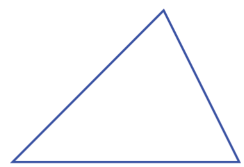


Thus we see that the two triangles have the same angles. So, their sides must be scaled by the same factor. Thus we also see that the third side of the small triangles is also $\frac{3}{4}$ of the third side of the large triangle.

Now even if the measures and the scale factor are different, we can reach the same conclusion using the above arguments.

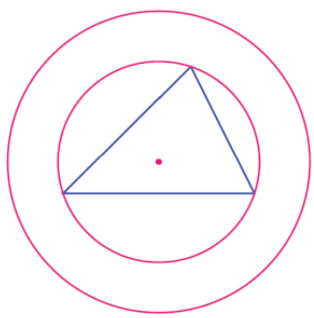
If two triangles have two of the sides scaled by the same factor and the included angles equal, then the third sides are also scaled by the same factor and the other two angles are also equal.

Using this result, we can scale a triangle without measuring any side or angle. For example, draw a triangle anyway you like.

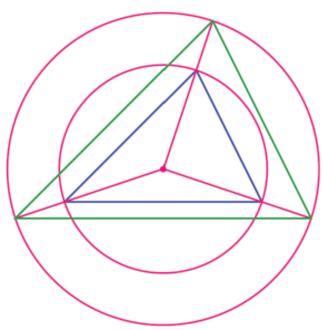


Let's see how the sides can be stretched one and a half times.

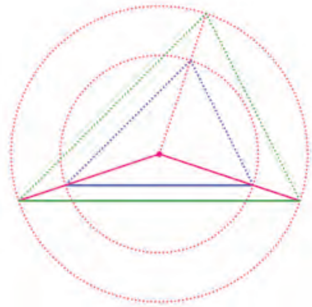
For this draw the circumcircle of the triangle and with the same centre, draw another circle of one and a half times the radius:



Now join the centre of the circles to the vertices of the triangle and extend them to meet the larger circle. Join these points to draw a larger triangle



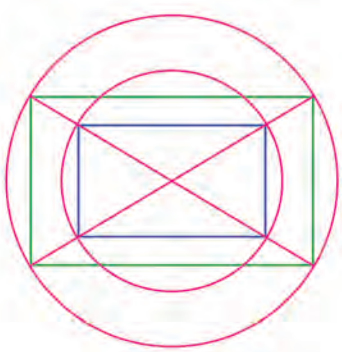
To see that the sides of the larger (green) triangle are one and a half times the sides of the smaller (blue) triangle, first focus on the triangles highlighted in the picture below:



The left and right sides of the larger triangle in this are one and a half times the left and right sides of the smaller triangle (why?) And the angle between them is the same for both the triangles. So, the bottom sides are also scaled by the same factor.

This means the bottom side of the green triangle drawn earlier is one and a half times the bottom side of the blue triangle drawn first. In the same way, we can see that the other sides of these triangles are also scaled by the same factor.

In a similar manner, we can enlarge or shrink a rectangle also:



In GeoGebra, create a slider *a* with Min=0. Draw triangle *ABC* and find its circumcentre *D*. (Use **Circle through 3 Points** tool to draw the circumcircle; select the **Midpoint** or **Centre** tool and click on the circle to get its centre). With *D* as centre draw a circle of radius 'a' times the circumradius (Give *a***AD* as radius). Using the **Ray** tool draw lines starting at *D* and passing through the vertices of the triangle. Mark the points where these meet the new circle and join them to make a triangle. Find the relation between the lengths of the sides of the two triangles.

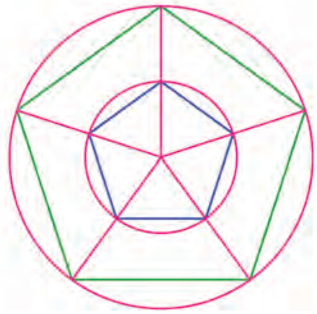
We cannot use this method for all quadrilaterals. For example, we cannot draw a circle through the four vertices of a parallelogram which is not a rectangle.

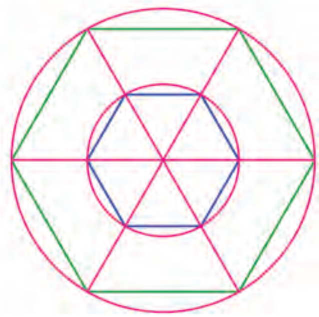
Right triangles

Just because two sides of a triangle are equal to two sides of another, the triangles may not be equal. Either the third sides must also be equal or the included angles must be equal. But two sides of a right triangle determine the third side, by the Pythagoras Theorem, so that two right triangles with the two sides with the same length would be equal.

In the same manner, two triangles with just two sides scaled by the same factor may not be similar; either the third sides must also be scaled by the same factor or the included angles must be equal. But in two right triangles, if two sides are scaled by the same factor, the third side also is scaled by this factor itself, and so the triangles are similar.

However, we can use this method to scale any regular polygon.





We summarize all the results we have discussed:

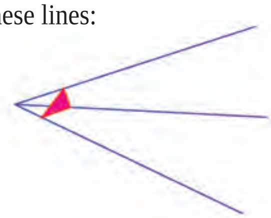
If two triangles have any one of the following relations, they also have the other two:

- Have the same angles
- Have all sides scaled by the same factor
- Have two sides scaled by the same factor and the included angle equal

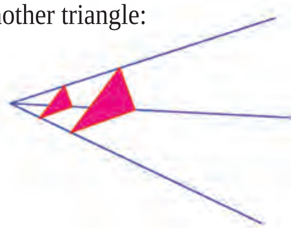
Triangles having any of these relations between them are said to be *similar.*

Projection

Join the vertices of a triangle to a point outside and extend these lines:

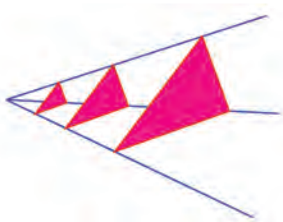


Draw a line parallel to the left side of the triangle, a little to the right of the triangle. Through the point where this meets the top blue line, draw another line parallel to the right side of the triangle. Through the point where it meets the middle blue line, draw a line parallel to the bottom side of the triangle. Now we get another triangle:



The two triangles have the same angles (why?) and so their sides are stretched by the same factor.

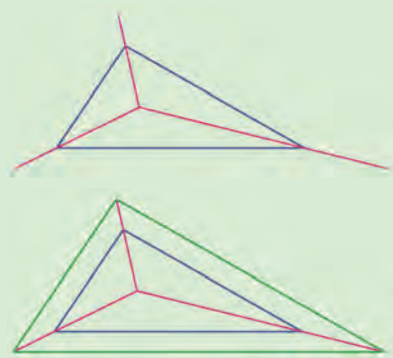
We can draw as many triangles as we want like this.



(1) Prove that if the perpendicular sides of two right triangles are scaled by the same factor, then the hypotenuses are also scaled by the same factor.

(2) Prove that if any two sides of two right triangles are scaled by the same factor, then the third side also is scaled by the same factor.

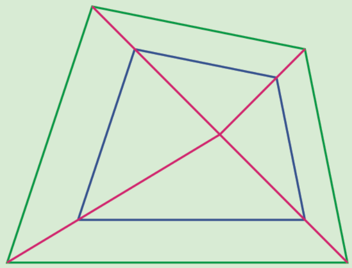
(3) Draw a triangle and mark a point inside it. Join this point to the vertices and extend each of them by half its original length. Join the end points of these lines to form another triangle;



Prove that the sides of the larger triangle are one and a half times the sides of the original triangle.

Draw triangle *ABC* in GeoGebra and mark a point *D* outside it. Using the **Ray** tool, join lines from *D*, passing through the vertices of the triangle. Mark a point *E* on the line *DA* and draw the line through *E*, parallel to *AB*. Mark the point *F* where this line meets *DB*. Similarly draw the line through *E*, parallel to *AC* and mark the point *G* where this line meets *DC*. Draw triangle *EFG*. Mark the angles of triangles *ABC* and *EFG*. Aren't they the same? Change the position of *E*. Doesn't the size of triangle *EFG* change? Are the sides of these two triangles scaled by the same factor?

(4) The vertices of a quadrilateral are joined to a point inside it and these line are extended by the same scale factor. The ends of these lines are joined to form another quadrilateral:



(i) Prove that the sides of the two quadrilaterals also are scaled by the same factor.

(ii) Prove that the angles of the two quadrilaterals are the same.

We can use the **Enlarge from Point** (**Dilate from Point**) tool in GeoGebra to draw similar triangles. Create a slider *a* with Min = 0. Draw a triangle and mark a point inside or outside it. Select the **Enlarge from Point** tool and click first on the triangle and then on the point. In the dialogue window, give *a* as the scale factor. We get another triangle similar to the first. Change the position of the point and the value of *a* and check. Instead of triangles, we can draw figures similar to any other figure in this way.

Project

Check whether the altitudes, medians and angle bisectors of similar triangles are scaled by the same factor as the sides.