1 Arithmetic Sequences

Number patterns

We have seen many number patterns:

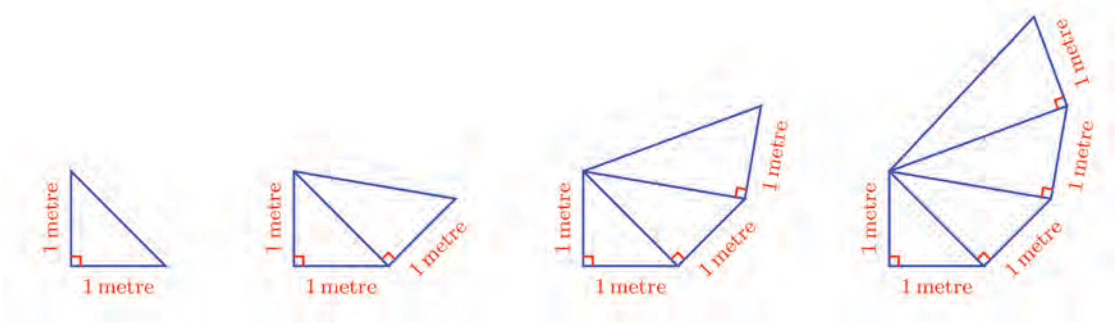
Natural numbers	1, 2, 3, ...
Even numbers	2, 4, 6, ...
Odd numbers	1, 3, 5, ...
Halves of natural numbers	$\frac{1}{2}$, 1, $1\frac{1}{2}$, ...
Squares of natural numbers	1, 4, 9, ...

A collection of numbers, which are ordered as the first, second, third and so on like this, according to some rule, is called a **number sequence.**

Such sequences often arise in computing various measurements. For example, if we order polygons according to the number of sides, as triangles, quadrilaterals, pentagons and so on and calculate the sum of their inner angles, we get the number sequence

180, 360, 540, ...

As another example, look at these pictures:



If we write down the length of the hypotenuse of the largest triangle in each picture, in metres, the sequence we get is that of square roots of natural numbers from 2

$$\sqrt{2}, \sqrt{3}, 2, \ldots$$

The same sequence can be described in many ways. For example, the sequence of natural numbers ending in 1 is

$$1, 11, 21, 31, \ldots$$

It can also be described as the sequence of natural numbers which leave remainder 1 on division by 10.

All kinds of sequences

The English word 'sequence' comes from the Latin word *sequent*, which means 'following'. In mathematics, we use this word to denote things arranged in definite positions as the first, second, third and so on. The things so arranged can be any objects, not necessarily numbers. For example, here's the sequence of regular polygons:

We can also have a sequence of polynomials like this:

$$1 + x, 1 + x^2, 1 + x^3, \ldots$$

The alphabetical arrangement of words of a language is also a sequence.

(1) We can make triangles by stacking dots:

Write the number of dots in each triangle. Calculate the number of dots needed to make the next three triangles in this pattern.

(2) From the sequence equilateral triangle, square, regular pentagon and so on of regular polygons, form the following sequences

Using GeoGebra, we can draw the sequence of regular polygons with a specified side.

First mark two points A, B. Then type this command in the Input Bar:

```
Sequence[Polygon[A,B,n],n,3,10]
```

This command is the instruction to change the value of the numbers n from 3 to 10, and draw regular polygon of n sides with AB as one side

To draw the polygons one by one, create an integer slider m and change the command as below:

```
Sequence[Polygon[A,B,n+2],n,1,m]
```

This means to draw regular polygons of sides 3, 4, 5, ... and so on, with *AB* as a side, as we change the value of m through 1, 2, 3, ... and so on using the slider

If we change n + 2 to 2n in the above command, what sort of polygons would we get? What if we change it to 2n + 1?

Number of sides	3, 4, 5, 6, ...
Sum of inner angles	
Sum of outer angles	
An inner angle	
An outer angle	

(3) Write the sequence of natural numbers which leave remainder 1 on division by 3, and the sequence of natural numbers which leave remainder 2 on division by 3.

(4) Write in ascending order, the sequence of natural numbers with last digit 1 or 6. Describe this sequence in two other ways.

(5) See these figures:

The first picture shows an equilateral triangle with the smaller triangle got by joining the midpoints of sides cut off. The second picture shows the same thing done on each of the three triangles in the first picture. The third picture shows the same thing done on the second picture.

(i) How many red triangles are there in each picture?

(ii) Taking the area of whole uncut triangle as 1, compute the area of a small triangle in each picture.

(iii) What is the total area of all the red triangles in each picture?

(iv) Write the first five terms of each of the three sequences got by continuing this process.

The numbers in a sequence can be generated using GeoGebra. To get the first ten numbers of the sequence 1, 4, 9, ... of perfect squares, type the command below in the Input Bar:

```
Sequence[n^2,n,1,10]
```

If we create an integer slider m and issue the command

```
Sequence[n^2,n,1,m]
```

then the number of squares in the sequence changes as we change m. The algebra of the sequence 2, 4, 8, ... is 2^n. To get this type

```
Sequence(2^n,n,1,m)
```

Circle division

If we choose any two points on a circle and join them with a line, it divides the circle into two parts:

If we choose three points instead and join them with lines, they divide the circle into four parts:

What if choose four points and join every pair?

How about five points?

How many parts do you expect to get by joining six points? Check your guess by actually drawing a picture.

Arithmetic sequences

The sequence of even numbers goes like this:

$$2, 4, 6, \ldots$$

What is the next number in this?

And the next after?

In general, to go from one number in this sequence to the next, we need only add 2.

Now let's look at the sequence of odd numbers:

$$1, 3, 5, \ldots$$

In this also, we need only add 2 to go from one number to the next.

The difference between these two is the number at which they start: the first sequence starts at 2 and proceeds by adding 2 successively; the second starts at 1 and proceeds the same way.

Now what if we take halves of natural numbers?

$$\frac{1}{2}, 1, 1\frac{1}{2}, \ldots$$

At what number does it start? And what number is successively added?

Let's look at some more sequences:

(i) The sequence of the sum of outer angles of polygons ordered according to the number of sides:

$$360, 360, 360, \ldots$$

Here we can say that the sequence starts at 360 and proceeds by adding 0 at every step

(ii) Water flows out of a tank containing 100 litres of water, at the rate of 5 litres each hour. If we calculate the water remaining in the tank in litres after each hour, we get the sequence

$$95, 90, 85, \ldots$$

This sequence starts at 95 and proceeds by subtracting 5 at every step.

Instead of saying subtracting 5, we can say adding –5.

Hasty conclusions

We discussed a problem on the number of parts of a circle got by joining points on it, in the section **Circle division**. For the number of points equal to 2, 3, 4, 5 we get the number of parts as 2, 4, 8, 16. What about 6 points? We tend to guess the number of parts as 32. Do we get this many parts if we actually draw such a picture?

If the points are equally spaced, we get 30 parts:

Otherwise, 31 parts

Either way the maximum number of parts is 31.

It can be proved that in general if we take n points on the circle and join all pairs, the maximum number of parts got is

$$\frac{1}{24}n(n-1)(n-2)(n-3) + \frac{1}{2}n(n-1) + 1$$

The interesting point is that in this expression and in 2^{n-1}, if we take $n = 2, 3, 4, 5$ we get the same numbers 2, 4, 8, 16. From $n = 6$ onwards, the numbers differ.

Let's have another look at the way the sequences seen now are formed:

Rule	Sequence
Start from 2 and successively add 2	2, 4, 6, ...
Start from 1 and successively add 2	1, 3, 5, ...
Start from $\frac{1}{2}$ and successively add $\frac{1}{2}$	$\frac{1}{2}$, 1, $1\frac{1}{2}$, ...
Start from 360 and successively add 0	360, 360, 360, ...
Start from 95 and successively add –5	95, 90, 85, ...

All such sequences have a common name

A sequence starting with a number and proceeding by adding one number again and again, is called an arithmetic sequence

The numbers in a sequence are called the *terms* of the sequence. For example, the terms of the sequence of half the natural numbers are $\frac{1}{2}$, 1, $1\frac{1}{2}$ and so on; more precisely, $\frac{1}{2}$ is the first term, 1 is the second term, $1\frac{1}{2}$ is the third term and so on.

Thus we can say this about an arithmetic sequence:

> In an arithmetic sequence, the same number is added to go from any term to the next

This can be put like this also:

> In an arithmetic sequence, the same number is subtracted to go from any term to the previous term.

This number, got by subtracting the previous term from any term, is called the **common difference** of the arithmetic sequence.

For example, the arithmetic sequence got by starting with 10 and adding 3 at each step is

10, 13, 16, 19, 22, ...

And the common difference of this sequence is 3

What if we start with 10 and subtract 3 at every step?

We get the sequence

$$10, 7, 4, 1, -2, \ldots$$

And the common difference is

$$7 - 10 = 4 - 7 = 1 - 4 = -2 - 1 = \cdots = -3$$

We often test whether a sequence is an arithmetic sequence by checking whether the number got by subtracting the previous term from a term is the same.

For example, let's look at the sequence of natural numbers which leave remainder 1 on division by 3:

$$1, 4, 7, \ldots$$

We can write these numbers in more detail like this:

$$(3 \times 0) + 1,\ (3 \times 1) + 1,\ (3 \times 2) + 1, \ldots$$

Thus if we take any two consecutive terms of this sequence, the first would be 1 added to 3 multiplied by a number and the second, 1 added to 3 multiplied by the next number. So, the first subtracted from the second would give 3.

So, this is an arithmetic sequence with common difference 3.

Next let's look at those natural numbers which do not leave a remainder 1 on division by 3

$$2, 3, 5, 6, \ldots$$

In this we have

$$3 - 2 = 1$$

$$5 - 3 = 2$$

Since these differences are not the same, the sequence is not an arithmetic sequence.

If the remainder on dividing a number by 3 is not 1, it can be either 0 or 2. Write down those leaving remainder 0 and those leaving remainder 2 separately. Are they arithmetic sequences?

Sequence rule

What is the next term of the sequence 3, 5, 7, ...?

Here, it's not said that the sequence is an arithmetic sequence. So, the next term may not be 9. For example, if the sequence is supposed to be that of odd primes, the next number is 11.

What's the moral of this?

From a few numbers written in a certain order, we cannot say for sure what the next numbers are.

For this, the rule of formulation of the sequence or the context in which it arises should be made clear.

(1) Check whether each of the sequences given below are arithmetic sequences. Give reasons also. Find the common differences of the arithmetic sequences:

(i) Natural numbers leaving remainder 1 on division by 4

(ii) Natural numbers leaving remainder 1 or 2 on division by 4

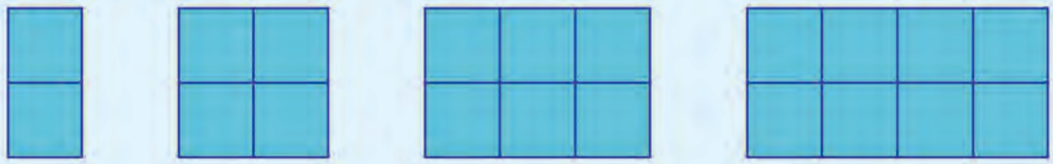
(iii) Squares of natural numbers

(iv) Reciprocals of natural numbers

(v) Powers of 2

(vi) Half of the odd numbers

(2) See these pictures:



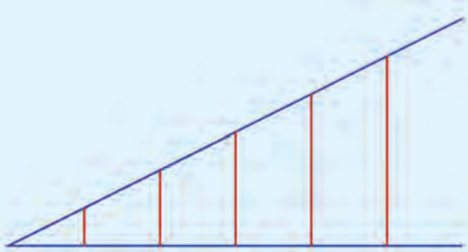
(i) How many small squares are there in each picture?

(ii) How many large squares?

(iii) How many squares in all in each picture?

If we continue the pattern of pictures, are the sequences above arithmetic sequences?

(3) In the picture below, the perpendiculars drawn from the bottom line are equally spaced.



Show that the sequence of the heights of the perpendiculars, on continuing this, form an arithmetic sequence.

(Hint: Draw perpendiculars from the top of each perpendicular to the next perpendicular)

Position and term

Can you write an arithmetic sequence with 1 and 11 as the first and second terms ?

Easy, isn't it ?

To get to 11 from 1, we must add 10; and if we continue adding 10, we get an arithmetic sequence:

1, 11, 21, 31, ...

Now another question: Can you write an arithmetic sequence with 1 and 11 as the first and the third terms ?

How do we find the common difference of such an arithmetic sequence ?

The second number is got by adding the common difference to 1; and we don't know this number. But we do know that adding the common difference again to this number gives the third number 11.

Sequence and remainder

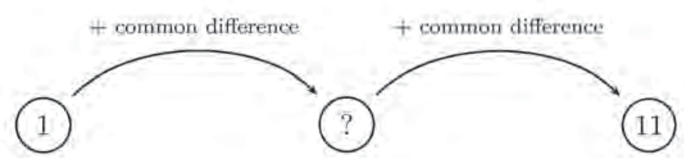
The even numbers 2, 4, 6, ... is an arithmetic sequence; so is the sequence 1, 3, 5, ... of odd numbers. And both have common difference 2.

Now even numbers are numbers divisible by 2 (that is, numbers which leave remainder 0 on division by 2) and the odd numbers are numbers which leave remainder 1 on division by 2.

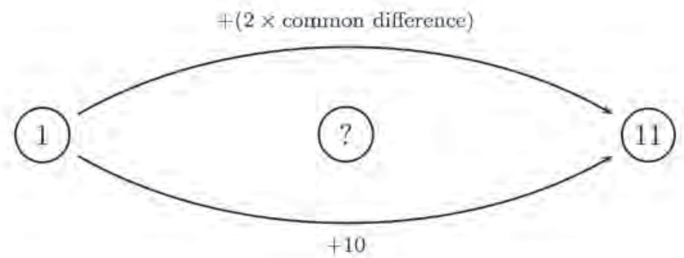
Similarly, we get three arithmetic sequences of natural numbers based on division by 3: those which leave remainder 0, 1 or 2. What is the common difference of each of these three sequences?

What if we divide by 4?

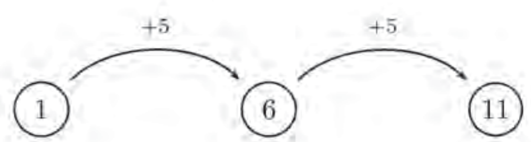
On the other hand, in any arithmetic sequence of natural numbers, the difference between any two terms is the common difference multiplied by a number. So, dividing these numbers by the common difference leave the same remainder (why?).



In other words, we get 11 by adding the common difference twice:



So, twice the common difference is 10 and so the common difference is 5



Now we can write the arithmetic sequence as

1, 6, 11, 16, 21, ...

In each of the arithmetic sequences below, some of the terms are not written, but indicated by ◯. Find out these numbers:

(i) 24, 42, ◯, ◯, ...   (ii) ◯, 24, 42, ◯, ...

(iii) ◯, ◯, 24, 42, ...   (iv) 24, ◯, 42, ◯, ...

(v) ◯, 24, ◯, 42, ...   (vi) 24, ◯, ◯, 42, ...

Let's look at another problem:

Calculate the first five terms of the arithmetic sequence with 3^{rd} term 37 and 7^{th} term 73.

We think like this:

(i) To go from the 3^{rd} term to the 7^{th} term, we must add the common difference $7 - 3 = 4$ times

(ii) The number added is $73 - 37 = 36$

(iii) So 4 times the common difference is 36

(iv) Common difference is $36 \div 4 = 9$

Next how do we compute the 1^{st} term?

(i) To go from the 3^{rd} term to the 1^{st} term, we must subtract the common difference $3 - 1 = 2$ times

(ii) The 1^{st} term is $37 - (2 \times 9) = 19$

Now can't we write the sequence ?

19, 28, 37, 46, 55, ...

The two terms in specific positions of some arithmetic sequences are given below. Write the first five terms of each:

(i)	(ii)	(iii)	(iv)	(v)
3^{rd} term 34	3^{rd} term 43	3^{rd} term 2	5^{th} term 8	5^{th} term 7
6^{th} term 67	6^{th} term 76	5^{th} term 3	9^{th} term 10	7^{th} term 5

Changes in position and terms

See this arithmetic sequence:

19, 28, 37, ...

In this,

1^{st} term is 19

2^{nd} term is 28

3^{rd} term is 37

....

We can write it like this:

Position	1	2	3	...
Term	19	28	37	...

How does the term change with the position in this?

When the position increases by 1, the term increases by 9

So, how do we compute the 10^{th} term of this sequence?

This can be done in several ways

We can start at the last term written above:

(i) From the 3^{rd} to the 10^{th}, position increases by $10 - 3 = 7$

(ii) To get the 10^{th} term from the 3^{rd}, we must add 7 times 9

(iii) The 10^{th} term is $37 + (7 \times 9) = 100$

Or we can start from the first term:

(i) From the 1^{st} to the 10^{th}, position increases by $10 - 1 = 9$

(ii) To get the 10^{th} term from the 1^{st}, we must add 9 times 9

(iii) The 10^{th} term is $19 + (9 \times 9) = 100$

Can you compute the 10^{th} term from the 2^{nd} like this?

Let's look at another problem:

The 4th term of an arithmetic sequence is 45, and the 5th term is 56. What is the first term?

To get the 5th term from the 4th term in this, the number added is 56 – 45 = 11. Since this is an arithmetic sequence, the same number 11 must be added to get any term from the previous term.

What we must compute is the 1st term

We can think like this:

(i) To get to the 1st term from the 4th term, the position must be decreased by 4 – 1 = 3

(ii) To get the 1st term from the 4th term, 11 must be subtracted 3 times

(iii) The 1st term is 45 – (3 × 11) = 12

Now we can write the sequence itself as

12, 23, 34, ...

In both these examples, the terms were increasing.

What about a sequence of decreasing terms?

For example, see this arithmetic sequence:

91, 82, 73, ...

In what all ways can we calculate its 10th term?

In this, the terms decrease as the positions increase. By how much?

For each increase in position by 1, the term decreases by 91 – 82 = 9

So, how do we compute the 10th term ?

(i) To get to the 10th term from the 3rd term, the position must be increased by 10 – 3 = 7

(ii) To get the 10th term from the 3rd term, 9 must be subtracted 7 times

(iii) The 10th term is 73 – (7 × 9) = 10

In such sequences as this, how do we compute a term from a term before it?

For example, see this problem:

The 4th term of an arithmetic sequence is 65 and its 5th term is 54. What is the first term?

In this sequence also, terms decrease as positions increase.

For each increase in position by 1, the term decreases by $65 - 54 = 11$

On the other hand, what happens for each decrease in position by 1?

The term increases by 11, right?

So how do we compute the 1st term?

(i) To get to the 1st term from the 4th term, the position must be decreased by $4 - 1 = 3$

(ii) To get the 1st term from the 4th term, 11 must be added 3 times

(iii) The 1st term is $65 + (3 \times 11) = 98$

(1) What is the 25th term of the arithmetic sequence 1, 11, 21, ... ?

(2) The 10th term of an arithmetic sequence is 46 and its 11th term is 51

(i) What is its first term?

(ii) Write the first five terms of the sequence

(3) What is the 21st term of the arithmetic sequence 100, 95, 90, ...?

(4) The 10th term of an arithmetic sequence is 56 and its 11th term is 51

(i) What is its first term ?

(ii) Write the first five terms of the sequence

Let's take a more detailed look at the relation between the changes in the positions and terms of an arithmetic sequence.

For example, consider this arithmetic sequence:

5, 15, 25, ...

We can write the connection between positions and terms like this:

Position	1	2	3	4	5	6	7	8	9	10	...
Term	5	15	25	35	45	55	65	75	85	95	...

In this, as the positions increase by 1, the terms increase by 10

What happens as the positions increase by 2?

Position	1	3	5	7	9	...
Term	5	25	45	65	85	...

What about the terms as the positions increase by 3 ?

On the other hand, what happens when the positions decrease?

- As the positions decrease by 1, the terms decrease by 10
- As the positions decrease by 2, the terms decrease by 20
- As the positions decrease by 3, the terms decrease by 30

Now let's look at an arithmetic sequence with decreasing terms:

Position	1	2	3	4	5	6	7	8	9	10	...
Term	50	45	40	35	30	25	20	15	10	5	...

What is the relation between the changes in positions and the changes in terms of this sequence?

- As the positions increase by 1, the terms decrease by 5
- As the positions increase by 2, the terms decrease by 10
- As the positions increase by 3, the terms decrease by 15

And when the positions decrease?

- As the positions decrease by 1, the terms increase by 5
- As the positions decrease by 2, the terms increase by 10
- As the positions decrease by 3, the terms increase by 15

In any other arithmetic sequence, the terms would be different, but the relation between the change in positions and the change in terms would be similar, right?

So what can we say in general ?

> In any arithmetic sequence, the change in terms is the product of the change in position and a fixed number

Using the ideas discussed in the chapter **Proportion** of the Class 9 textbook, we can state this as follows:

> In any arithmetic sequence, the change in terms is proportional to the change in position

Now look at this problem:

The 3^{rd} term of an arithmetic sequence is 12 and the 7^{th} term is 32. What is its 15^{th} term?

First let's write the position change and the term change from the 3^{rd} to the 7^{th}

Position change	7 – 3	=	4 increase
Term change	32 – 12	=	20 increase

From this, we can see that the terms increase by $20 \div 4 = 5$ as the positions increase by 1

We want the 15^{th} term. As the position change from 7^{th} to the 15^{th}, there's an increase of $15 - 7 = 8$; and so the term increases by $8 \times 5 = 40$.

Thus the 15^{th} term is $32 + 40 = 72$

There's another way of doing this

(i) As the position increases by 4, the term increases by 20

(ii) When the position increases by $8 = 2 \times 4$, the term increases by $2 \times 20 = 40$

(iii) 15^{th} term is $32 + 40 = 72$

How do we compute the 5^{th} term like this?

(i) 3^{rd} term is 12

(ii) 5^{th} term is 2 positions ahead of the 3^{rd} term

(iii) As the position increases by 4, the term increases by 20

(iv) When the position increases by $2 = \frac{1}{2} \times 4$, the term increases by $\frac{1}{2} \times 20 = 10$

(v) 5^{th} term is $12 + 10 = 22$

Let's look at another problem:

The 4^{th} term of an arithmetic sequence is 81 and the 6^{th} term is 71. What is its 20^{th} term?

As in the first problem, let's first write down the changes in positions and terms:

Position change	6 – 4	=	2 increase
Term change	81 – 71	=	10 decrease

What we want is the 20^{th} term; that is 20 – 6 = 14 positions ahead of the 6^{th} term. The calculations can be done like this:

(i) When the position increases by 2, the term decreases by 10

(ii) When the position increases by 14 = 7 × 2 the term decreases by 7 × 10 = 70

(iii) The 20^{th} term is 71 – 70 = 1

(1) The 3^{rd} term of an arithmetic sequence is 15 and the 8^{th} term is 35

(i) What is its 13^{th} term?

(ii) What is its 23^{rd} term?

(2) The 5^{th} term of an arithmetic sequence is 21 and the 9^{th} term is 41

(i) What is its first term?

(ii) What is its 3^{rd} term?

(3) The 4^{th} term of an arithmetic sequence is 61 and the 7^{th} term is 31

(i) What is its 10^{th} term?

(ii) What is its first term?

(4) The 5^{th} term of an arithmetic sequence is 10 and the 10^{th} term is 5

(i) What is its 15^{th} term?

(ii) What is its 25^{th} term?

We can use the fact that the term change in an arithmetic sequence is proportional to the position change, to check whether a number is a term of an arithmetic sequence.

For example, let's look at an earlier example of an arithmetic sequence:

19, 28, 37, ...

We have seen that the 10^{th} term of this sequence is 100

Is 1000 a term of this sequence?

As we move from the first to the second term of this sequence, the term increases by 28 – 19 = 9. So, the relation between position difference and term difference can be written like this:

Position change	1	2	3	...
Term change	9	2×9	3×9	...

Thus any term difference is a multiple of 9.

So, to check whether 1000 is a term of this sequence, we need only check whether $1000 - 19 = 981$ is a multiple of 9.

Since we have

$$981 \div 9 = 109$$

We can see that 1000 is actually the 110^{th} term of this sequence.

Now check whether 10000 is a term of this sequence, and if so find its position.

(1) Is 101 a term of the arithmetic sequence 13, 24, 35, ...? What about 1001?

(2) In the table below, some arithmetic sequences are given and two numbers against each. Check whether the numbers are terms of the respective sequences:

Sequence	**Numbers**	**Yes/No**
11, 22, 33, ...	123	
	132	
12, 23, 34, ...	100	
	1000	
21, 32, 43, ...	100	
	1000	
$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots$	3	
	4	
$\frac{3}{4}, 1\frac{1}{2}, 2\frac{1}{4}, \ldots$	3	
	4	

(3) In the table above, find the position of the numbers that are terms of the respective sequences.

Term connections

We have noted that if we know the terms in two specified position of an arithmetic sequence, then we can calculate all its terms.

If we know just one term and its position, can we say anything about the sequence ?

For example, suppose we know that the first term of an arithmetic sequence is 5, can we say anything more about the sequence ?

The common difference can be any number; and so the sequence can be continued in whatever way we want.

What if we know that the second term is 5 ?

In this case also, we can take any number as the common difference and proceed.

Let's try few numbers as the common difference and the terms just before and after the 2nd term 5 (that is, the 1st and the 3rd terms)

Common difference 1 : 4, 5, 6

Common difference 2 : 3, 5, 7

Common difference $\frac{1}{2}$: $4\frac{1}{2}, 5, 5\frac{1}{2}$

Common difference $\sqrt{2}$: $5 - \sqrt{2}, 5, 5 + \sqrt{2}$

Common difference -1 : 6, 5, 4

Is there any relation between these numbers?

What is the sum of the first and last numbers in each case?

And the sum of all three?

Why does this happen?

Whatever be the common difference, the first term is this number subtracted from 5, and the third term is this number added to 5

So, adding these two numbers is in effect, adding two 5's, right?

Next, to add all three terms, the second term 5 must be added once more, which gives the sum of three 5's

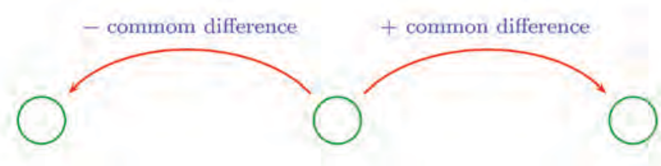
What if we take some number other than 5 as the second term?

In that case also, wouldn't the sum of the first and the third terms be twice that number?

And what about the sum of all three terms?

Now what if we take some other term instead of the second?

The term just before it is the common difference subtracted from it; and the term just after is the common difference added to it:



So, their sum is the twice number first chosen; that is the middle term

What about the sum of all three terms?

So, what can we say in general?

(i) In an arithmetic sequence, the sum of the terms just before and just after a term is twice this term.

(ii) In an arithmetic sequence, the sum of a term and the terms just before and just after it, is three times this term.

If we take a term of an arithmetic sequence and the terms just before and after that, they form three consecutive terms of the sequence, right?

So the second statement above can also put like this:

In any arithmetic sequence, the sum of three consecutive terms is three times the middle term.

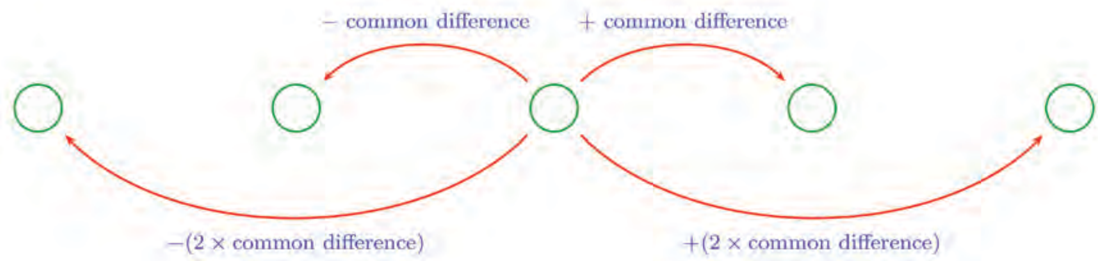
We can also state this in reverse:

In any arithmetic sequence, the middle term of three consecutive terms is one-third of their sum

For example if the sum of the 5^{th}, 6^{th} and the 7^{th} terms of an arithmetic sequence is 30, then the 6^{th} term is $\frac{1}{3} \times 30 = 10$

Now in addition to the terms just before and after a term, what if we also take terms two positions behind and two positions ahead ?

Moving two positions behind means twice the common difference is subtracted from the term and moving two positions ahead means adding twice the common difference to the term:



We have already seen that the sum of the terms just before and just after a term is twice this term.

What about the sum of the terms two positions behind and two position ahead?

One of these is the twice common difference subtracted from the term first chosen and the other is twice the common difference added to this term. So their sum also is twice this term, isn't it?

What about the sum of all five terms?

The sum of the first and the fifth is twice the third number (which is the number in the middle) and so is the sum of the second and fourth. So the sum of all five is 2+2+1 = 5 times the third number.

Now what if we start from a term and take pairs of terms one, two and three positions behind and after?

What can we say in general?

(i) In an arithmetic sequence, the sum of the two terms, at the same distance behind and ahead a term, is twice this term

(ii) In an arithmetic sequence, the sum of a term and the consecutive terms at the same distance behind and ahead, is the product of this term and the number of terms

For example, if the 10^{th} term of an arithmetic sequence is 25, what all sums can we compute using the first statement above ?

- The sum of the 9^{th} and 11^{th} terms is $2 \times 25 = 50$
- The sum of the 8^{th} and 12^{th} terms is $2 \times 25 = 50$

What other sums of terms can be calculated to be 50?

Using the second statement above, what all sums can we calculate?

- The sum of the terms in positions 9, 10, 11 is $3 \times 25 = 75$
- The sum of the terms in positions 8, 9, 10, 11, 12 is $5 \times 25 = 125$

What other sums can we compute ?

Did you notice another thing here? If we take a term and consecutive terms at the same distance before and ahead, we get an odd number of terms, right? And the term we started with, is the term in the middle of all these.

So, the second statement above can be stated in a different way:

The sum of an odd number of consecutive terms of an arithmetic sequence is the product of the middle term and the number of terms

(1) The 4^{th} term of an arithmetic sequence is 8.

(i) Find the sum of the pairs of terms given below:

(a) 3^{rd} and 5^{th}

(b) 2^{nd} and 6^{th}

(c) 1^{st} and 7^{th}

(ii) What is the sum of the 3^{rd}, 4^{th} and the 5^{th} terms?

(iii) What is the sum of the 5 terms from the 2^{nd} to the 6^{th}?

(iv) What is the sum of the 7 terms from the 1^{st} to the 7^{th}?

(2) The common difference of an arithmetic sequence is 2 and the sum of the 9^{th}, 10^{th} and 11^{th} terms is 90. Calculate the first three terms of the sequence.

Now look at this problem:

The first term of an arithmetic sequence is 10 and the sum of the first 5 terms is 250. Can you find the sequence?

Since the sum of the first 5 terms is 250, the 3^{rd} term (the middle term) can be computed as 50 (How?).

Since the 1^{st} term is 10 and the 3^{rd} term is 50, the common difference can be found as 20
So, the sequence is 10, 30, 50, ...

(1) Write three arithmetic sequences with the sum of the first 7 terms as 70.

(2) The sum of the first 3 terms of an arithmetic sequence is 30 and the sum of the first 7 terms is 140.

(i) What is the 2^{nd} term of the sequence?

(ii) What is the 4^{th} term of the sequence?

(iii) What are the first three terms of the sequence?

(3) The sum of the first five terms of an arithmetic sequence is 150, and the sum of the first ten terms is 550

(i) What is the third term of the sequence?

(ii) What is the eighth term of the sequence?

(iii) Write the first three terms of the sequence

(4) The sum of the 11^{th} and 21^{st} terms of an arithmetic sequence is 80. What is the 16^{th} term?

(5) The angles of a pentagon are in arithmetic sequence

(i) If the angles are written according to their magnitude, what would be the third angle?

(ii) If the smallest angle is 40°, what are the other angles?

(iii) Can the smallest angle be 36°?

We have seen that just by knowing the term at a specific position, we can compute the sum of many terms.

Let's look at a different kind of problem:

If the sum of the 2^{nd} and 5^{th} terms of an arithmetic sequence is known to be 35, can we say anything more about the sequence?

We can take any number as the 2^{nd} term. For example, let's take the 2^{nd} term as 10.

Then the 5^{th} term must be $35 - 10 = 25$

Now we can compute the common difference as $(25 - 10) \div 3 = 5$ and then write the first few terms of the sequence as

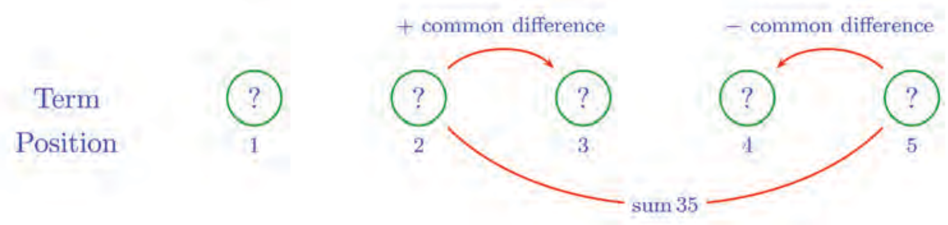
$$5, 10, 15, 20, 25, 30, \ldots$$

Did you notice something about the sequence?

sum of the 3rd and 4th terms = 15+ 20 = 35

sum of the 1st and 6th terms = 5 + 30 = 35

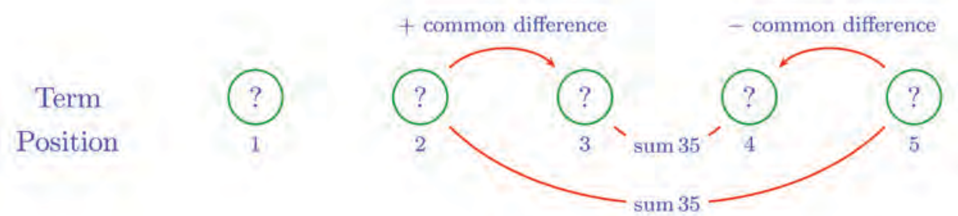
Take some other number as the 2nd term and calculate these sums. Why are they always 35? Let's draw a picture showing this:



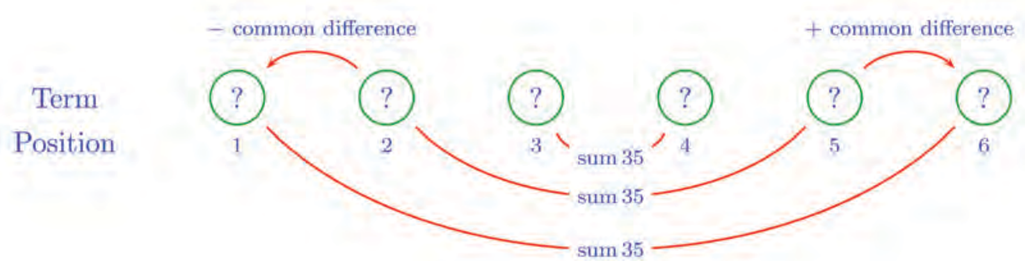
To go from the 2nd term to the 3rd term, we must add the common difference

To go from the 5th term to the 4th term, we must subtract the common difference

So, the sum of the 3rd and 4th terms is also 35, isn't it?



What about the sum of the 1st and 6th terms?



Similarly, if the sum of the 4th and 8th terms is known to be 45, what other sums can we calculate?

- Moving one position forward from the 4th and one position backward from the 8th, the sum of the 5th and the 7th terms is 45
- Moving one position backward from the 4th and one position forward from the 8th, the sum of the 3rd and the 9th terms is 45

What other terms have the same sum 45?

Positions	4, 8	5, 7	3, 9	...	...
Sum of terms	45	45	45	...	...

Similarly, if the sum of the 2nd and 6th terms is said to be 30, the sum of the terms at what other pairs of positions can we say is 30 ?

What can we say in general about this ?

In an arithmetic sequence, if one position is increased and another position is decreased by the same amount, the sum of the terms at these positions do not change

The sum of two numbers doesn't change, if one number is increased and the other decreased by the same amount, right? On the other hand, if the sum of a pair of numbers is equal to the sum of another pair, then one of the numbers has increased and the other decreased by the same amount.

So, the above fact can be stated like this also:

In an arithmetic sequence, if the sum of two positions is equal to the sum of other two positions, then the sum of the terms at each pair is the same

So, if we know that the sum of the first four terms of an arithmetic sequence is 100, what other things can we say about the sequence?

The first four terms can be split into two pairs with the same sum of positions:

- 1st and 4th
- 2nd and 3rd

Since the sum of all four terms is 100, the sum of the terms at each pair above must be 50. That is

- The sum of the 1st and 4th terms is 50
- The sum of the 2nd and 3rd terms is 50

Now suppose that we also know that the 1st term is 10

Then we can proceed like this

- The 4th term is 50 – 10 = 40
- The 1st term is 10 and the 4th term is 40; so, 3 times the common difference is 40 – 10 = 30
- The common difference is 30 ÷ 3 = 10
- The first four terms of the sequence are 10, 20, 30, 40

(1) Write four arithmetic sequence with sum of the first four terms 100

(2) The 1st term of an arithmetic sequence is 5 and the sum of the first 6 terms is 105. Calculate the first six terms of the sequence.

(3) The sum of the 7th and the 8th terms of an arithmetic sequence is 50. Calculate the sum of the first 14 terms.

(4) Write the first three terms of each of the arithmetic sequences given below:

(i) The 1st term is 30 and the sum of the first three terms is 300

(ii) The 1st term is 30 and the sum of the first four terms is 300

(iii) The 1st term is 30 and the sum of the first five terms is 300

(iv) The 1st term is 30 and the sum of the first six terms is 300