5 Second Degree Equations

Square problems

Let's start with a problem:

> When a square was enlarged, extending each side by 1 metre, its perimeter became 36 metres. What was the length of a side of the original square?

Easy, isn't it?

The length of a side of the new square is 36 ÷ 4 = 9 metres and so, the length of a side of the original square was 9 – 1 = 8 metres.

What if we change the question to this:

> When a square was enlarged, extending each side by 1 metre, its area became 36 square metres. What was the length of a side of the original square?

What is the length of a side of the new square?

$\sqrt{36} = 6$ metres, right?

So, the length of a side of the original square was 6 – 1 = 5 metres

(1) When each side of a square was reduced by 2 metres to make a smaller square, its area became 49 square metres. What was the length of a side of the original square?

(2) There is a 2 metre wide path around a square ground. The area of the ground and path together is 1225 square metres. What is the area of the ground alone?

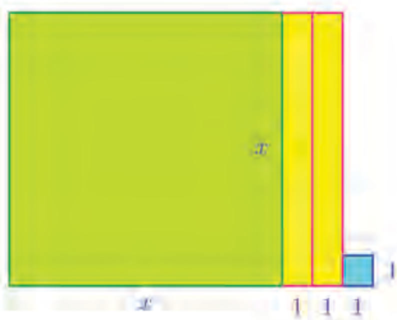
See this picture:

A green square, two yellow rectangles of the same height and a small blue square joined together. The width of each yellow rectangle and a side of the blue square are both 1 metre. And the area of the whole figure is 100 square metres.

We want to calculate the length of a side of the green square.

Difficult to do directly, isn't it?

Let's try algebra, taking the length of a side of the green square as x:



Then the total area of the figure is

$$x^2 + x + x + 1 = x^2 + 2x + 1$$

The total area is given to be 100 square metres. So, we can translate the problem to algebra like this:

If $x^2 + 2x + 1 = 100$, then what is x ?

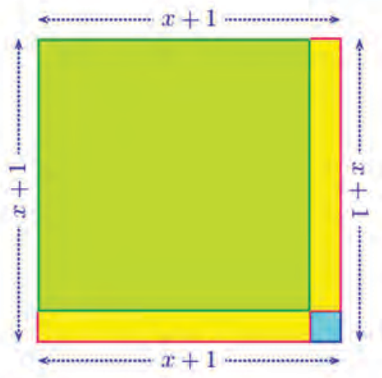
Does the expression $x^2 + 2x + 1$ look familiar?

Remember the equation

$$(x + 1)^2 = x^2 + 2x + 1$$

seen in the lesson **Square Identities** of the Class 8 textbook?

This can also be seen by rearranging the pieces of our figure:



So we can rephrase our problem like this:

If $(x + 1)^2 = 100$, then what is x?

Now it is easy to see that $x + 1 = 10$ and so $x = 9$

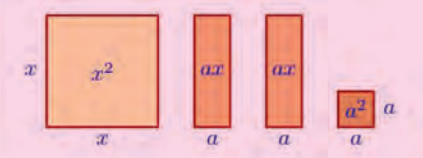
That is, the lengths of a side of of the green square is 9 metres.

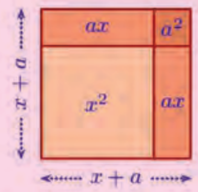
Squares - algebra and geometry

For any two numbers x and a, we have

$$x^2 + 2ax + a^2 = (x + a)^2$$

If x and a are positive numbers, we can translate this algebraic identity to a geometrical picture:





(1) 1 added to the product of two consecutive even numbers gave 289. What are the numbers?

(2) 9 added to the product of two consecutive multiples of 6 gave 729. What are the numbers?

(3) The first few terms of the arithmetic sequence 9, 11, 13, ... were added and then 16 added to the sum, to get 256. How many terms were added?

Square completion

Look at this problem:

One side of a rectangle is 2 metres longer than the other and its area is 224 square metres. What are the lengths of the sides?

Let's first translate this to algebra. If we take the length of the shorter side as x metres, then the length of the longer side is $x + 2$ metres and the area is $x (x + 2) = x^2 + 2x$ square metres.

Now we have the algebraic problem:

If $x^2 + 2x = 224$, then what is x?

What next?

Remember how we rewrote $x^2 + 2x + 1$ as $(x + 1)^2$ and went ahead? In this problem, we have only $x^2 + 2x$

Why not just add a 1?

So, we can continue like this:

(i) Since $x^2 + 2x = 224$, we have $x^2 + 2x + 1 = 224 + 1 = 225$

(ii) That is, $(x + 1)^2 = 225$

(iii) Since $(x + 1)^2 = 225$ we have $x + 1 = \sqrt{225} = 15$

(iv) Since $x + 1 = 15$, we have $x = 14$

Thus we get the shorter side of the rectangle as 14 metres.

So, the longer side is 16 metres

Suppose we change this problem slightly like this:

One side of a rectangle is 20 metres longer than the other and its area is 224 square metres. What are the lengths of the sides?

The algebraic form of the problem changes like this:

If $x^2 + 20x = 224$ then what is x ?

Here also, if we add 1, the number on the right side of the equation becomes $225 = 15^2$, but the expression on the left of the equation becomes $x^2 + 20x + 1$

Can we put it in the form $(x + a)^2$?

Whatever number we take as a,

$$(x + a)^2 = x^2 + 2ax + a^2$$

In our equation, we have $x^2 + 20x$; that is $20x$ in the place of $2ax$ in the general equation.

So, what if we take a as 10?

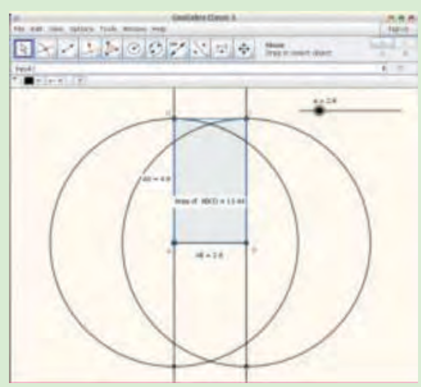
$$(x + 10)^2 = x^2 + 20x + 100$$

In our problem, we have $x^2 + 20x = 224$. How about adding 100, following what we saw just now?

$$\begin{aligned} x^2 + 20x &= 224 \\ x^2 + 20x + 100 &= 324 \\ (x + 10)^2 &= 324 \\ x + 10 &= \sqrt{324} = 18 \\ x &= 8 \end{aligned}$$

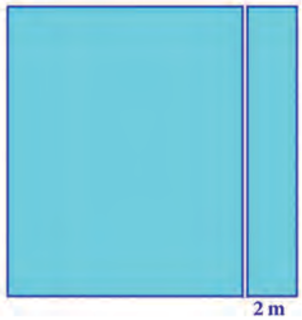
Thus we can see that the lengths of the sides of the rectangle are 8 metres and 28 metres

We can draw rectangles with one side 2 longer than the other. For that, first create a slider a with minimum value 0. Draw a line AB of length a, and draw perpendiculars through its endpoints. Draw circles of radius a + 2, centres at A and B. Mark the points C and D, where the perpendiculars meet the circle. Draw the rectangle joining these four points. Now we can hide the perpendiculars and the circles. Mark the area of the rectangle. At what value of a does the area becomes 224?



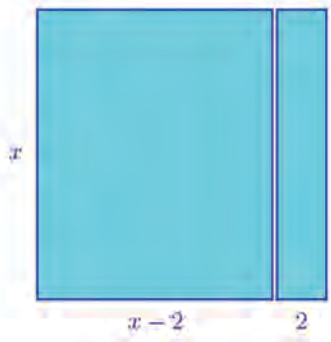
Here is another problem on rectangles:

From a square, a rectangle of width 2 metres is cut off.



The area of the remaining rectangle is 99 square metres. What was the length of a side of the original square?

Taking the length of a side of the square as x metres, the lengths of the sides of the remaining rectangle are x metres and $x - 2$ metres;



And the area of the remaining rectangle is $x(x - 2) = x^2 - 2x$ square metres.

So, the algebraic form of the problem is this:

If $x^2 - 2x = 99$, then what is x ?

Can you change $x^2 - 2x$ to a squared expression, as we did with $x^2 + 2x$?

Recall another identity from Class 8:

$$x^2 - 2x + 1 = (x - 1)^2$$

Now can't we find the x in our problem?

$$\begin{aligned} x^2 - 2x &= 99 \\ x^2 - 2x + 1 &= 100 \\ (x - 1)^2 &= 100 \\ x - 1 &= 10 \\ x &= 11 \end{aligned}$$

The length of the side of the original square is 11 metres.

Another problem:

One of the perpendicular sides of a right triangle is 5 centimetres more than the other. The area of the triangle is 12 square centimetres. What are the lengths of these sides?

Taking the length of the shorter of the perpendicular sides as x, the length of the other side is $x + 5$. What about the area?

So, translating the given information into algebra, we get

$$\frac{1}{2}x(x+5) = 12$$

This we can write as

$$x^2 + 5x = 24$$

What number added to $x^2 + 5x$ changes it to the form $x^2 + 2ax + a^2$?

Write

$$x^2 + 5x = x^2 + \left(2 \times \frac{5}{2}\right)x$$

and think what number is to be added to change it to a squared expression

$$x^2 + \left(2 \times \frac{5}{2}\right)x + \left(\frac{5}{2}\right)^2 = \left(x + \frac{5}{2}\right)^2$$

So, let's add $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ to both sides of our equation:

$$x^2 + 5x + \frac{25}{4} = 24 + \frac{25}{4}$$

That is,

$$\left(x + \frac{5}{2}\right)^2 = \frac{121}{4}$$

From this, we get

$$x + \frac{5}{2} = \sqrt{\frac{121}{4}} = \frac{11}{2}$$

and then

$$x = \frac{11}{2} - \frac{5}{2} = 3$$

Thus the lengths of the perpendicular sides of the triangle are 3 centimetres and 8 centimetres.

Now look at this problem:

A rectangle is to be constructed with perimeter 100 metres and area 525 square metres. What should be the lengths of its sides?

The sum of the length and breadth of the rectangle is 50 centimetres. So, taking the length of one side as x metres, the length of the other side is $50 - x$ metres.

And the area is $x(50 - x) = 50x - x^2$ square metres

So, we can rewrite our problem like this:

$$\text{If } 50x - x^2 = 525, \text{ then what is } x\text{ ?}$$

We cannot write $50x - x^2$ as a squared expression. So, let's rewrite the equation in another form. The number x^2 subtracted from the number $50x$ gives 525. So, the reverse subtraction gives the negative, -525. Thus the problems can be written like this:

$$\text{If } x^2 - 50x = -525, \text{ then what is } x\text{ ?}$$

Now we have to add a number to $x^2 - 50x$ to make it a squared expression. What number should we add?

$$x^2 - 50x + 625 = (x - 25)^2$$

So, we can solve our problem like this:

$$\begin{aligned} x^2 - 50x &= -525 \\ x^2 - 50x + 625 &= -525 + 625 = 100 \\ (x - 25)^2 &= 100 \\ x - 25 &= 10 \\ x &= 35 \end{aligned}$$

Thus the lengths of the sides of the rectangle are 35 metres and 15 metres.

This problem can be done in a different way

The length of the side of a square of perimeter 100 metres is 25 metres and its area is 625 square metres. But the area of the rectangle in our problem is 525 square metres. So, it is not a square. So the length of all the sides is not 25 metres.

If both sides of this rectangle are longer than 25 metres, the perimeter would be greater than 100 metres; and if both are shorter than 25 metres, then the perimeter would be less than 100 metres. So, one side must be longer than 25 metres and the other less than 25 metres.

Since the sum of these lengths must be 50 (why?), the increase in one length and the decrease in the other must be the same.

If that is taken as x metres, then the lengths of the sides are $25 + x$ metres and $25 - x$ metres.

And the area $(25 + x)(25 - x)$ square metres

So, the problem in algebra is this:

$$\text{If } (25 + x)(25 - x) = 525, \text{ then what is } x?$$

We can write $(25 + x)(25 - x)$ as

$$(25 + x)(25 - x) = 625 - x^2$$

(The section **Difference of squares** of the lesson **Square Identities** in the Class 8 textbook).

Using this, the problem can be changed to this

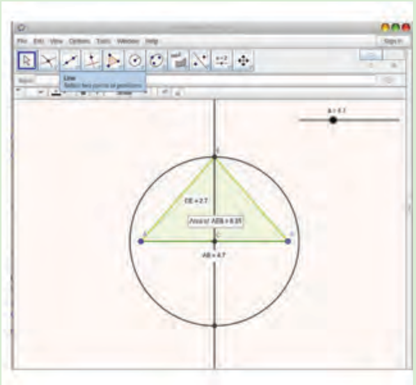
$625 - x^2 = 525$, then what is x ?

From this, we can see that $x^2 = 625 - 525 = 100$ and so $x = 10$.

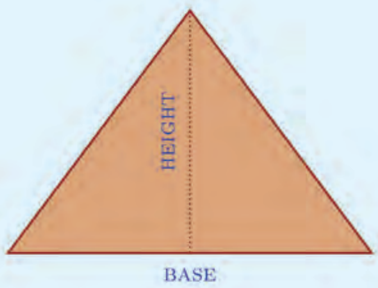
This means the lengths of the sides are

$25 + 10 = 35$ metres and $25 - 10 = 15$ metres

In GeoGebra, make a slider a with Min:2 and Max:10 and draw a line of length a. Mark its midpoint and perpendicular bisector. With the midpoint as centre, draw the circle of radius a−2. Mark the point where the bisector meets the circle and complete the triangle. Mark the area of the triangle. Now change the value of a using the slider. When is the area 12?



(1) An isosceles triangle like the one shown below must be made:



The height must be 2 metres less than the base and the area should be 12 square metres. What should be the lengths of the sides of the triangle?

(2) One side of a right triangle is one centimetre shorter than twice the side perpendicular to it and the hypotenuse is one centimetre longer than twice the length of this side. What are the lengths of the sides?

(3) A pole 2.6 metres long leans against a wall. The foot of the pole is 1 metre away from the wall. When the foot of the pole was pushed a little away, the top end slid down by the same length. How much was the foot moved?

(4) The product of two consecutive odd numbers is 195. What are the numbers?

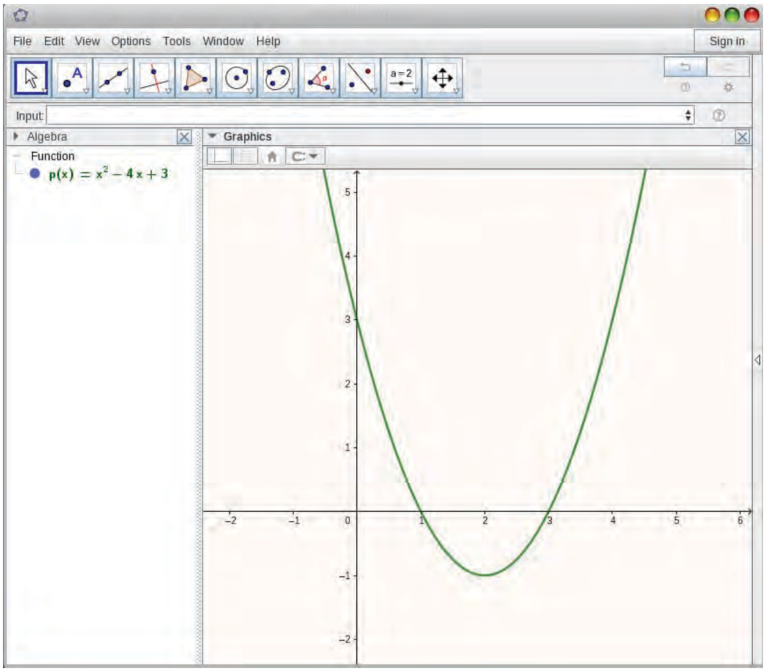
(5) How many terms of the arithmetic sequence 5, 7, 9, ... must be added to get 140 as the sum ?

Two solutions

We have seen graphs of second degree polynomials in the lesson **Polynomial Pictures** of the Class 9 textbook (The section **Second degree polynomials**).

For example, using GeoGebra, we get this as the graph of the polynomial

$$p(x) = x^2 - 4x + 3$$



The graph cuts the horizontal line at the points marked 1 and 3, right?

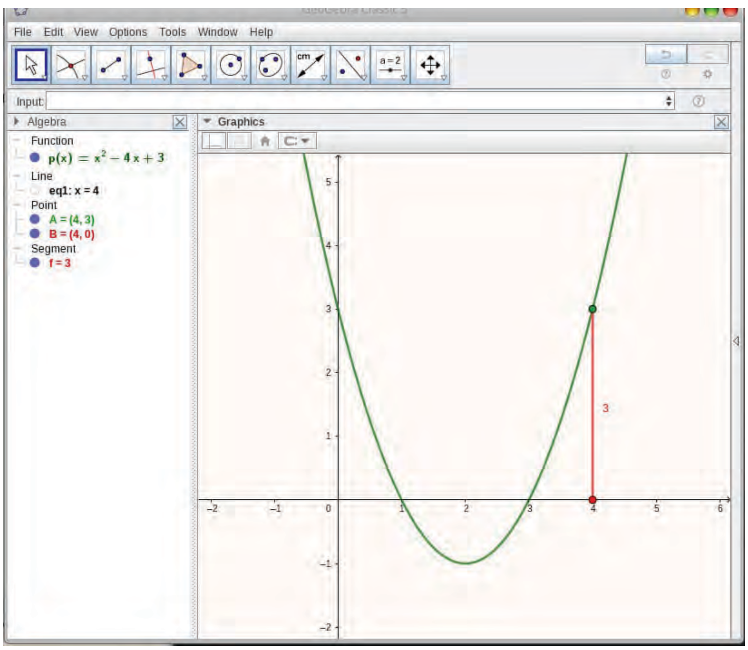
So, what are the numbers $p(1)$ and $p(3)$?

In general, for any number x on the horizontal line, the number $p(x)$ is the height from this point to the graph of $p(x)$.

For example

$$p(4) = 4^2 - (4 \times 4) + 3 = 3$$

and we can see from the picture that the height from the point 4 on the horizontal line to the graph of the polynomial is 3



This height from the points 1 or 3 on the horizontal line is zero. Are $p(1)$ and $p(3)$ equal to zero?

Let's check:

$$p(1) = 1^2 - (4 \times 1) + 3 = 1 - 4 + 3 = 0$$
$$p(3) = 3^2 - (4 \times 3) + 3 = 9 - 12 + 3 = 0$$

Can we compute the numbers x for which $p(x) = 0$ algebraically, without drawing its graph?

The question is this:

$$\text{If } x^2 - 4x + 3 = 0, \text{ then what is } x\ ?$$

Can we solve this problem like we did the earlier ones?

Can we add a number to $x^2 - 4x + 3$ to make it a squared expression?

We know that

$$x^2 - 4x + 4 = (x-2)^2$$

This we can write like this

$$(x^2 - 4x + 3) + 1 = (x-2)^2$$

Now we can find the x we want:

$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ x^2 - 4x + 3 + 1 &= 0 + 1 \\ x^2 - 4x + 4 &= 1 \\ (x-2)^2 &= 1 \\ x - 2 &= 1 \\ x &= 3 \end{aligned}$$

That is $p(3) = 0$, as seen earlier

But we have seen earlier that $p(1) = 0$ also. In other words, $x = 1$ is also a solution to the equation $x^2 - 4x + 3 = 0$.

Why didn't we get this answer now?

Let's look again at the steps in finding the solution through algebra.

At one stage in this process, we got

$$(x-2)^2 = 1$$

and from that we found $x - 2 = 1$

If the square of a number is 1, can we say that the number itself is 1?

What is the square of -1?

$$(-1)^2 = (-1) \times (-1) = 1$$

(The section **Oscillation** of the lesson **Negative Numbers** in the Class 9 textbook).

So in our problem, from the equation

$$(x - 2)^2 = 1$$

We can only say that

$$x - 2 = 1 \text{ or } x - 2 = -1$$

In this

$$\text{If we take } x - 2 = 1 \text{ we get, } x = 1 + 2 = 3$$
$$\text{If we take } x - 2 = -1 \text{ we get, } x = -1 + 2 = 1$$

Thus using the algebraic method also, we get both the numbers we can take as x to get $p(x) = 0$

This raises a question: if in the problems done so far, would we have got another solution, had we taken the negative root?

For example, let's look at a problem about rectangles done earlier: finding the sides of a rectangle with one side 2 metres longer than the other and area 224 square metres.

To solve this, we took x metres as the length of the shorter side and got the equation $(x + 1)^2 = 225$; then we took $x + 1 = 15$ and found the length of the shorter side as 14 metres.

If we just look at the algebra only, we should also consider $x + 1 = -15$, so that $x = -16$

But here, x is the length of the side of a rectangle and so is a positive number. Thus the solution $x = -16$ is not possible in our problem on rectangles.

Let's look at another rectangle problem done earlier: rectangle of perimeter 100 metres and area 525 square metres.

In this, taking the length of one side as x metres, we got the equation $(x - 25)^2 = 100$; then took $x - 25 = 10$ to get the length of one side as $(25 + 10) = 35$ metres and the length of the other side as $50 - 35 = 15$ metres

Suppose we take the negative square root?

We get $x - 25 = -10$ and from this $x = 15$; that is, the length of one side as 15 metres and the length of the other as $50 - 15 = 35$ metres.

Thus in this problem, we get the same rectangle, whether we take the positive or negative square root.

In general, when we translate a practical problem to an algebraic problem and think only about the mathematics, we may get more than one solution. Some of these or sometimes even all of them may not be suitable answers to the practical problem.

So what we do is to find all the solutions algebraically and then choose only those solutions suitable to the context.

Now look at this problem:

> How many terms of the arithmetic sequence 99, 97, 95, ... starting from the first must we add to get 900 as the sum?

We can find the n^{th} term of this sequence as $101 - 2n$ and the sum of n terms as

$$101n - n(n + 1) = 100n - n^2$$

So, the problem can be translated to algebra as

If $100n - n^2 = 900$, what is n ?

We can proceed like this:

$$\begin{aligned} 100n - n^2 &= 900 \\ n^2 - 100n &= -900 \\ n^2 - 100n + 50^2 &= -900 + 2500 \\ (n - 50)^2 &= 1600 \end{aligned}$$

What do we get from this ?

1600 is the square of both 40 and -40

So from the above equation we get

$$n - 50 = 40 \text{ or } n - 50 = -40$$

and this gives two solutions

$$n = 90 \text{ or } n = 10$$

Thus whether we add the first 10 terms of the sequence or the first 90 terms of the sequence, we get the same sum 900.

(1) The product of a number and 2 added to it is 168. What are the numbers?

(2) Find two numbers with sum 4 and product 2

(3) Consider the arithmetic sequence 55, 45, 35, ...

   (i) How many terms of this, starting from the first, must be added to get 175 as the sum?

   (ii) How many terms of this, starting from the first, must be added to get 180 as the sum?

   (iii) How many terms of this, starting from the first, must be added to get 0 as the sum?