IES / ISS EXAM, 2010

Serial No. 0021

C–HLR–K–TB

STATISTICS—II

Time Allowed : Three Hours | *Maximum Marks : 200*

INSTRUCTIONS

*Candidates should attempt **FIVE** questions in **ALL** including Questions No. **1** and **5** which are compulsory. The remaining **THREE** questions should be answered by choosing at least **ONE** question each from Section A and Section B.*

The number of marks carried by each question is indicated against each.

*Answers must be written only in **ENGLISH**.*

(Symbols and abbreviations are as usual.)

Any essential data assumed by candidates for answering questions must be clearly stated.

SECTION—A

1. Attempt any 5 parts of the following :—

   (a) Samples of sizes n_1 and n_2 are drawn from two population having means μ_1 and μ_2 respectively with a common variance σ^2. Find the BLUE of $l_1\mu_1 + l_2\mu_2$ and obtain its variance. 8

   (b) State and prove Gauss-Markoff theorem stating all basic assumptions and model requirements. 8

   (c) Consider four independent random variables with $E(y_1) = E(y_2) = \theta_1 + \theta_2$; $E(y_3) = E(y_4) = \theta_1 + \theta_3$; $V(y_i) = \sigma^2$, $i = 1, 2, 3, 4$.

   Determine the condition for estimability of a linear parametric function $\ell'\theta = \sum_{i=1}^{4} \ell_i \theta_i$. Also obtain a solution of the normal equations and the sum of squares due to error. 8

   (d) If $X_1, X_2, \ldots X_n$ are iid from N(0, 1). Investigate the behaviour of the following estimators in terms of unbiasedness, consistency, efficiency and sufficiency :

   (i) $T_1 = \frac{X_1 + 2X_2}{3}$, (ii) $T_2 = \frac{\sum_{i=1}^{n} X_i}{n}$,

   (iii) $T_3 = \frac{\sum_{i=1}^{n} X_i}{n+1}$. 8

(e) If

$$f(x/\theta) = \exp\left[\frac{1}{x}\{\theta A'(\theta) - A(\theta)\} - A'(\theta) + S(x)\right]$$

is a distribution indexed by parameter θ, then obtain the Maximum likelihood estimator of θ. 8

(f) Discuss the following estimation procedures :

(i) Pivotal method of constructing confidence interval.

(ii) Bootstrap method of finding revised estimates. 8

2. (a) Let X_1, X_2, X_3 be three uncorrelated random variables having common variance σ^2. If $E(X_1) = \theta_1 + \theta_2$, $E(X_2) = 2\theta_1 + \theta_2$, $E(X_3) = \theta_1 + 2\theta_2$. Show that the least squares estimates $\hat{\theta}_1, \hat{\theta}_2$ satisfy the equations :

$$6\hat{\theta}_1 + 5\hat{\theta}_2 = x_1 + 2x_2 + x_3$$

$$5\hat{\theta}_1 + 6\hat{\theta}_2 = x_1 + x_2 + 2x_3.$$

Obtain :

(i) $V(\hat{\theta}_1), V(\hat{\theta}_2)$ and $Cov(\hat{\theta}_1, \hat{\theta}_2)$

(ii) the Error sum of squares. 10

(b) Let $X_1, X_2, \ldots, X_n$ denote a random sample from a population with mean μ and variance $\sigma^2 (< \infty)$.

Find the BLUE of μ. Using appropriate examples show that this estimator may attain Cramer-Rao lower bound for some distribution, while for some other distribution, there exists an unbiased estimator which is better than the BLUE. 10

(c) State and prove factorization theorem for sufficiency. Hence or otherwise investigate the sufficiency of the estimator, $T = \frac{1}{6}(X_1 + 2X_2 + 3X_3)$ for θ in Bernoulli distribution. 10

(d) Give the statement of Bhattacharya bounds. Investigate the existence of Bhattacharya bound for θ if :

$$f(x; \theta) = \theta(1 - \theta)^x, \; x = 0, 1, 2, \ldots\ldots$$

Further verify whether there exists any MVB estimator or not ? Compare the estimates. 10

3. (a) For a model $E(y) = A\theta$, $D(y) = \sigma^2 I$, describe the estimation space and error space and find the least squares estimator for θ. Show that the columns of A and $Y - A\hat{\theta}$ are orthogonal. 10

(b) Define a one-way analysis of variance model, stating all its assumptions. Prepare the complete ANOVA table with the corresponding Expected SS for error and total. 10

(c) Let $(X_1, X_2, ..., X_m)$ and $(Y_1, Y_2, ..., Y_n)$ be two independent samples from $N(\mu_1, \sigma^2)$, $N(\mu_2, \sigma^2)$ respectively where μ_1, μ_2 and σ^2 are all unknown. Obtain a $100(1 - \alpha)\%$ confidence interval for $\mu_1 - \mu_2$. 10

(d) Show that a consistent solution of a likelihood equation is asymptotically normal under certain conditions to be stated. 10

4. (a) Let y_i, $i = 1, 2, ... n$ be independent random variables with $E(y_i) = \alpha + \beta x_i$; $V(y_i) = \sigma^2$, where $x_1, x_2, ..., x_n$ are given. Obtain the estimators of α, β and the variances of the estimators. 10

(b) Let $(X_1, X_2, ..., X_n)$ be a random sample from $N(\mu, \sigma^2)$, σ^2 is unknown. Use pivotal method to find a $100(1 - \alpha)\%$ confidence interval for μ. 10

(c) Distinguish between Uniformly minimum variance unbiased (UMVU) and Minimum variance bound (MVB) estimator. Show that UMVU estimator is always unique, if it exists. 10

(Contd.)

(d) If $\overline{X}_1$ is the mean of a random sample of size n from $N(\mu, \sigma_1^2)$ and $\overline{X}_2$ is the mean of another random sample of size n from $N(\mu, \sigma_2^2)$, and the two samples are independent then show that :

(i) $p\overline{X}_1 + (1-p)\overline{X}_2, 0 \le p \le 1$ is unbiased for μ, and

(ii) the variance of the estimator is minimum when $p = \dfrac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$. 10

SECTION—B

5. Attempt any **5** :—

(a) Distinguish between :

(i) Simple and composite hypothesis

(ii) Randomized and non-randomized tests

(iii) Null and alternative hypothesis

(iv) Prior and posterior distribution. 8

(b) Given a random sample from $N(\mu, 1)$, propose a test for $H_0 : \mu = 10$ against $H_1 : \mu = 11$. Obtain the critical region of the test and power of the test. 8

(c) Discuss the steps involved in setting the control limits for R-charts when the population range R is unknown. 8

(d) Obtain the moment generating function of a multivariate normal distribution $N_p(\mu, \Sigma)$. 8

(e) If $X \sim$ Poisson (λ) and if λ assumes a prior distribution gamma (α, β), obtain a Bayes estimate for λ. 8

(f) Explain sequential probability ratio test procedure. Obtain OC and ASN functions for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, if X follows a Bernoulli distribution with parameter θ. 8

6. (a) Define :

(i) Exponential family of distributions

(ii) Uniformly most powerful test

(iii) Unbiased test

(iv) Similar test. 10

(b) If two samples of size n_1 and n_2 are drawn from two normal populations with different variances, develop a test for :

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ versus } H_1 : \sigma_1^2 \neq \sigma_2^2. \qquad 10$$

(c) Find the distribution of sample correlation coefficients r_{ij}'s $(i < j = 1, 2 \ldots. \rho)$ when the corresponding population correlation coefficients, $\rho_{ij} \neq 0$ $(i \neq j = 1, 2 \ldots. \rho)$ in sampling from a p-variate normal distribution. 10

(d) Distinguish between LTPD and AOQL. Design a sampling plan for inspecting large batches (N large) which has a 95% chance of accepting batches with 0·5% defective and a 5% chance of accepting batches with 5% defective. 10

7. (a) A coin is tossed for which probability θ of getting head is either $\frac{1}{3}$ or $\frac{2}{3}$. We are asked to decide which value of θ is true. The coin is tossed once and the loss function is $L(\theta, d)^2 = (\theta - d)^2$. Find :

   (i) All possible decision rules

   (ii) Risk for all decision rules

   (iii) Minimax decision rules. 10

(b) Define Mahalanobis D^2-Statistic. Establish a suitable relationship with T^2-Statistics. Hence or otherwise obtain the null distribution of D^2-Statistics. 10

(c) If $X \sim N_p(\mu_1, \Sigma)$ and $X = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}$, where

$$X^{(1)} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix}, \text{ and } X^{(2)} = \begin{pmatrix} x_{q+1} \\ \vdots \\ x_p \end{pmatrix}.$$

Obtain $E[X^{(1)} \mid X^{(2)}]$. 10

(d) Let X be a single observation from a probability function p(x) which is positive as the non-negative integers and zero elsewhere. For testing

$$H_0 : p(x) = \frac{e^{-1}}{x!} \text{ versus } H_1 : p(x) = \frac{1}{2^{x+1}},$$ obtain the critical region and show that the test is unbiased. 10

8. (a) Let $X_{(1)} < X_{(n)}$ be the extreme Statistics in a random sample of size n from a rectangular distribution over $(\theta, \theta + 1)$. To test $H_0 : \theta = 0$ against $H_1 : \theta > 0$, the test is rejected iff $X_{(1)} > 1$ or $X_{(n)} > c$, where c is a constant. Determine c so that the test has size α and show that the test is unbiased. 10

(b) Show that SPRT terminates with probability one. 10

(c) Large batches of screws are subject to a single sampling plan with $n = 60$, $c = 2$. If the process average $\overline{P} = 0{\cdot}01$, does this lot accept batches of high quality $(p \le \overline{p})$ with high probability ? If the proportion of defectives in a batch is 0·05, what is the chance of accepting the batch ? 10

(d) Define a Wishart distribution. State and prove its reproductive property. 10